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Benjamin B. Kennedy
Gettysburg College

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Abstract
We consider state-dependent delay equations of the form $x'(t)=f(x(t-d(x(t))))$ where d is smooth and f is smooth, bounded, nonincreasing, and satisfies the negative feedback condition $xf(x)x\neq 0$. We identify a special family of such equations each of which has a ``rapidly oscillating" periodic solution p. The initial segment p0 of p is the fixed point of a return map R that is differentiable in an appropriate setting.

We show that, although all the periodic solutions p we consider are unstable, the stability can be made arbitrarily mild in the sense that, given $\epsilon>0$, we can choose f and d such that the spectral radius of the derivative of R at p0 is less than $1+\epsilon$. The spectral radii are computed via a semiconjugacy of R with a finite-dimensional map.

Keywords
periodic solution, stability, state-dependent delay, oscillating chemical reactions, delay differential equations

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A STATE-DEPENDENT DELAY EQUATION WITH NEGATIVE FEEDBACK AND "MILDLY UNSTABLE" RAPIDLY OSCILLATING PERIODIC SOLUTIONS

Benjamin B. Kennedy

Department of Mathematics
Gettysburg College
Gettysburg, PA 17325-1484, USA

Abstract. We consider state-dependent delay equations of the form

$$x'(t) = f(x(t - d(x(t))))$$

where d is smooth and f is smooth, bounded, nonincreasing, and satisfies the negative feedback condition $xf(x) < 0$ for $x \neq 0$. We identify a special family of such equations each of which has a "rapidly oscillating" periodic solution p. The initial segment p_0 of p is the fixed point of a return map R that is differentiable in an appropriate setting.

We show that, although all the periodic solutions p we consider are unstable, the stability can be made arbitrarily mild in the sense that, given $\epsilon > 0$, we can choose f and d such that the spectral radius of the derivative of R at p_0 is less than $1 + \epsilon$. The spectral radii are computed via a semiconjugacy of R with a finite-dimensional map.

1. **Introduction.** In this paper we consider the real-valued autonomous state-dependent delay equation

$$x'(t) = f(x(t - d(x(t)))). \tag{1}$$

We assume that f is smooth and bounded and satisfies the negative feedback condition $xf(x) < 0$ for all $x \neq 0$; and that d is smooth and bounded, satisfies $d(0) = 1$, and assumes values in $(0, 1]$. We shall also confine ourselves to the case that, if $x(t)$ is any solution of (1) defined on $[-1, \infty)$,

$$\frac{d}{dt}(t - d(x(t))) > 0 \text{ for any } t > 0. \tag{2}$$

Condition (2) allows us to define a nonincreasing oscillation speed for solutions that agrees with the usual oscillation speed in the constant delay case $d(x) \equiv 1$. We shall provide careful definitions of these notions — and, for that matter, of solutions of (1) — below.

For the general theory of state-dependent delay equations, we refer to the review article [3]. Results on equations of the form (1) and its generalizations include [15] (on the solution semiflow in a "C^1 framework"), [2] (on differentiability properties of solutions), and [8] and [10] (on existence of periodic solutions).

In this paper we exhibit some "rapidly oscillating" periodic solutions of some very special versions of (1). More particularly, given any $\alpha \in (0, 1)$, we shall present a

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family of delay functions d_γ, $\gamma \in [0, 1-\alpha]$, and a fixed nonincreasing feedback function f such that, for each $\gamma \in [0, 1-\alpha]$, the equation

$$x'(t) = f(x(t - d_\gamma(x(t)))) \tag{3}$$

has a rapidly oscillating periodic solution p^γ. We shall show, roughly speaking, that each p^γ is unstable but that γ can be chosen so that the instability is mild — more particularly, so that the initial segment p_0^γ of p^γ is the fixed point of a return map whose derivative at p_0^γ has spectral radius $(2/(2-\alpha))^2$. We shall take f close, in an appropriate sense, to the step function $-$sign. For our state space we use the C^1 solution manifold described in [15] and [3]. For assessing the stability of our periodic solutions we apply a semiconjugacy approach presented in [6].

Remark 1. One motivation for this work is simply that results concerning the stability of periodic solutions of state-dependent delay equations are still few, and the approach we use here provides a simple way to assess stability in certain (admittedly highly restricted) cases.

Another motivation for this work is as follows. In the constant-delay case $d(x) \equiv 1$ and with f strictly decreasing, rapidly oscillating periodic solutions of (1) must be unstable. (This result was conjectured in [5] and is proven in [11]; see [13] for an earlier proof with additional assumptions.) On the other hand, in the constant-delay case and with an instantaneous damping term added — as in (4) below — stable rapidly oscillating periodic solutions have been exhibited for non-monotonic f in [4] and [12]. The question of whether (1) (with no damping) can have stable rapidly oscillating periodic solutions in the constant-delay case with non-monotonic f is still open [1]. While we emphasize that the periodic solutions we find here are indeed unstable (and our feedback functions f only nonincreasing, rather than strictly decreasing), our results here indicate, roughly speaking, how state-dependency in the delay can mitigate instability. In our view, this finding heightens the interest of whether some instance or alteration of (1), with monotonic feedback, might admit stable periodic solutions that could also be reasonably called "rapidly oscillating."

Remark 2. It appears that one can construct examples similar to the ones we present here for equations of the form (1) with instantaneous damping added:

$$x'(t) = -\mu x(t) + f(x(t - d(x(t)))), \quad \mu > 0. \tag{4}$$

That is, we can construct "rapidly oscillating" periodic solutions whose instability is arbitrarily mild. We have not carried through all the details; the $\mu = 0$ case we present here seems to be considerably simpler.

In Section 2 we recall some general theory for equation (1), present the framework for assessing stability of periodic solutions, and state our main result. As mentioned above, this result concerns the existence of a family of equations with particular properties; in Section 3 we prove our main result by exhibiting such a family.

2. **General theory and main results.**

We consider the equation

$$x'(t) = f(x(t - d(x(t)))) \tag{5}$$

and assume the following throughout:

(H1): $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, bounded with bounded derivative, and satisfies the negative feedback condition $xf(x) < 0$ for all $x \neq 0$;

(H2): $d : \mathbb{R} \to (0, 1]$ is continuously differentiable with bounded derivative.

By a *solution* of (5) we mean either a continuously differentiable function $x : [-1, \infty) \to \mathbb{R}$ that satisfies (5) for all $t > 0$ or a continuously differentiable function $x : \mathbb{R} \to \mathbb{R}$ that satisfies (5) for all $t \in \mathbb{R}$.

Hypotheses (H1) and (H2) are enough to guarantee that we can study equation (5) in the "C^1 solution manifold" framework described in [15] and Section 3 of [3]. We now recall the results that we shall need from this framework, and introduce some notation.

We write $C = C[-1, 0]$ for the Banach space of continuous real-valued functions on $[-1, 0]$, equipped with the sup norm

$$\|\phi\|_0 = \sup_{s\in[-1,0]} |\phi(s)|.$$

If x is a continuous real-valued function whose domain includes $[t-1, t]$, we write x_t for the member of C given by $x_t(s) = x(t+s)$, $s \in [-1, 0]$.

We write $C^1 = C^1[-1, 0]$ for the Banach space of continuously differentiable real-valued functions on $[-1, 0]$, equipped with the norm

$$\|\phi\| = \|\phi\|_0 + \|\phi'\|_0.$$

Throughout, we write $DG[q]$ for the derivative of the map G at the point q.

We write

$$\mathcal{D} = \{ \ \phi \in C^1 \ : \ \phi'(0) = f(\phi(-d(\phi(0)))) \ \}.$$

Under our hypotheses, $\mathcal{D}$ is a codimension-1 submanifold of C^1. We shall endow $\mathcal{D}$ with the metric it inherits as a topological subspace of C^1, with derivatives of functions on $\mathcal{D}$ defined via its submanifold structure: if G is a differentiable map defined on $\mathcal{D}$, the derivative of G at $\phi \in \mathcal{D}$ is a bounded linear map on the tangent space $T_\phi\mathcal{D}$ to $\mathcal{D}$ at ϕ. $T_\phi\mathcal{D}$ is defined by

$$T_\phi\mathcal{D} = \{ \ \psi \in C^1 \ : \ \psi'(0) = Dg[\phi]\psi \ \},$$

where $g : C^1 \to \mathbb{R}$ is the function defined by $g(\phi) = f(\phi(-d(\phi(0))))$.

With the exception of the fact that solutions of (5) are defined for all $t > 0$ — which is an easy consequence of the boundedness of f — the following is just a restatement of Theorem 3.2.1 and Proposition 3.3.1 in [3], specialized to our situation.

Proposition 1 (Solutions of (5)). *There is a uniquely defined continuous solution semiflow $F : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ for (5). The maps $F(t, \cdot) : \mathcal{D} \to \mathcal{D}$ are completely continuous (i.e. map bounded sets to precompact sets) for all $t \geq 1$. F is continuously differentiable on $(1, \infty) \times \mathcal{D}$.*

(The complete continuity of the maps $F(t, \cdot)$ for $t \geq 1$ hinges on the equicontinuity of $F(t, \phi)$ and its derivative $F(t, \phi)'$. In [3] this is established via a Lipschitz estimate, with respect to the sup norm, on the map $g(\phi) = f(\phi(-d(\phi(0))))$. We provide such a Lipschitz estimate in the proof of Lemma 3.9 below.)

If $x : [-1, \infty) \to \mathbb{R}$ is a solution of (5), we shall call x_0 the *initial condition* of x, and x the *continuation of* x_0 *as a solution of* (5).

Remark 3. Under hypotheses (H1) and (H2), solutions can be defined with initial conditions in a larger space of Lipschitz functions. Such solutions, though, will eventually flow into $\mathcal{D}$, so considering the $\mathcal{D}$ to be the state space does not sacrifice any substantial dynamical information about solutions of (5).

As mentioned in Section 1, we shall focus on instances of (5) where condition (2) is satisfied:

$$\frac{d}{dt}(t - d(x(t))) > 0 \text{ for any } t > 0 \text{ and any solution } x \text{ of (5).}$$

If x is a solution of (5) and $t > 0$, then

$$\frac{d}{dt}(t - d(x(t))) = 1 - d'(x(t))x'(t).$$

We therefore arrive at the very simple criterion that (2) is satisfied if

$$\sup_{y\in\mathbb{R}} |d'(y)| \sup_{y\in\mathbb{R}} |f(y)| < 1, \tag{6}$$

which will be sufficient for our purposes. (See Lemma 1.2 of [8] for a weaker version of (2) that holds more generally. For more sophisticated conditions guaranteeing (2) for equations of the form (4), see [7] and [9].)

We now define the "oscillation speed" of a solution. Our definition is similar to that in [7], and to the usual definitions of oscillation speed in the constant-delay case.

Definition 2.1. a) Suppose that x is a solution of (5), and that $x(z) = 0$. We call z a *proper zero* if $x(z + \epsilon)$ and $x(z - \epsilon)$ are of strictly opposite signs for all sufficiently small $\epsilon > 0$.

b) Given a solution x of (5) and $t \geq 0$, we define $os(x_t)$, the *oscillation speed of x at t*, to be the number of proper zeros of x on the interval $(\tau - d(0), \tau)$, where

$$\tau = \inf\{s \geq t \ : \ s \text{ is a proper zero of } x \},$$

provided this infimum exists.

The following result is familiar (the ideas are the same as in the constant delay case).

Lemma 2.2. *Suppose that (H1), (H2), and (2) are satisfied, and that x is a solution of (5). If $t \geq 0$, x_t has finitely many zeros, and $os(x_t)$ is defined, then $os(x_t)$ is even. Moreover, in this case the following hold:*

- *if $s \geq t$ and $os(x_s)$ is defined, then $os(x_s) \leq os(x_t)$ (oscillation speed is nonincreasing); and*
- *if $s \geq t$ and $os(x_s)$ is not defined, then $x(\tau) \to 0$ as $\tau \to \infty$.* □

We now describe how we will assess stability of periodic solutions of (5). We continue to assume that (H1) and (H2) hold, and we write $F : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{D}$ for the solution semiflow described in Proposition 1. We begin by discussing the return maps we are interested in; the ideas are familiar. Let us write $\mathcal{D}_0 = \{\phi \in \mathcal{D} : \phi(0) = 0 \}$. Suppose that $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (5), translated so that $p(0) = 0$ and $p'(0) > 0$. Suppose also that p has period dividing τ_0, with $\tau_0 > 1$. Then there is, by the implicit function theorem, a relatively open neighborhood U of p_0 in $\mathcal{D}_0$ and a unique differentiable function $\tau : U \to (1, \infty)$ such that $\tau(p_0) = \tau_0$ and

$$F(\tau(x_0), x_0) \in \mathcal{D}_0 \text{ for all } x_0 \in U.$$

The map $R : U \to \mathcal{D}_0$ given by $R(x_0) = F(\tau(x_0), x_0)$ is called a *return map*. This map is differentiable and completely continuous.

It is readily seen that R can in fact be extended to a relatively open neighborhood about p_0 in $\mathcal{D}$ — otherwise put, an open neighborhood about p_0 of initial conditions in $\mathcal{D}$ have forward orbits that eventually flow into $\mathcal{D}_0$. Thus we see that the dynamics of the solution semiflow near p — in particular, whether solutions near p diverge from p or (up to translation) remain near it — are captured by the dynamics of the return map $R : U \subset \mathcal{D}_0 \to \mathcal{D}_0$ near p_0. We therefore may regard p as stable if p_0 is a stable fixed point of R, and unstable otherwise. This is the approach we take here: in particular, we shall obtain information, for a special class of equations (5), about the spectrum of the derivative $DR[p_0]$ of R at p_0. (We refer to [16] for a description, in the constant-delay case, of the intimate connection between the spectrum of $DR[p_0]$ and the so-called Floquet multipliers of p.)

To compute the spectrum of $DR[p_0]$ we shall exploit a semiconjugacy between R and a finite-dimensional map. We shall use the framework presented in [6], which we now recall in a specialized and condensed form. The basic ideas involved have been exploited by many authors; see [6] for references and a fuller discussion.

Suppose that $\mathcal{B}$ is a Banach space, and that $Y \subset X$ are subsets of $\mathcal{B}$ (equipped with the subspace topology). Let $p_0 \in Y$ be a point such that, locally about p_0, both X and Y have the structure of Banach submanifolds of $\mathcal{B}$, with the tangent space to Y at p_0 a subspace of the tangent space to X at p_0: $T_{p_0}Y \subset T_{p_0}X$. Suppose that $U \subset X$ is a relatively open subset of X that contains the point p_0.

Finally, assume that V is a Banach space.

We now impose the following hypotheses.

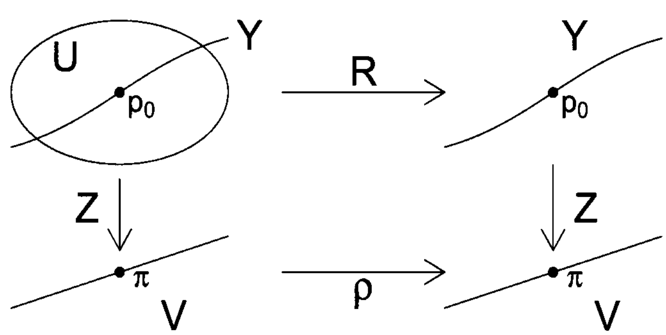
(I): There is a completely continuous and continuously differentiable map $R : U \to X$ with the features that $R(p_0) = p_0$ and $R(U) \subset Y$.

(II): There is a continuous, open, and continuously differentiable map $Z : Y \to V$ with the feature that, given $x, y \in U \cap Y$, $Z(x) = Z(y) \implies R(x) = R(y)$.

(III): $DZ[p_0] : T_{p_0}Y \to V$ is surjective, and $\ker DZ[p_0] \subset \ker DR[p_0]$.

(IV): There is a completely continuous, continuously differentiable map $\rho : Z(U \cap Y) \to V$ with the feature that $\rho(Z(x)) = Z(R(x))$ for all $x \in U \cap Y$.

See the figure below.



The following proposition (and its proof) is a version of various results in Section 2 of [6].

Proposition 2. *Assume that X, Y, U, and V are as above, and that (I)–(IV) hold. Then:*

- $\pi := Z(p_0)$ *is a fixed point of* ρ;
- *The nonzero spectrum of* $DR[p_0]$ *is equal to the nonzero spectrum of* $D\rho[\pi]$.

Proof. That ρ fixes π follows immediately from (IV):

$$\rho(\pi) = \rho(Z(p_0)) = Z(R(p_0)) = Z(p_0) = \pi.$$

For notational brevity, let us write $A = T_{p_0}X$ and $B = T_{p_0}Y$. Since R maps U into Y, we have $DR[p_0](A) \subset B \subset A$. By the chain rule, for all $y \in B$ we have $D\rho[\pi]DZ[p_0]y = DZ[p_0]DR[p_0]y$. Since ρ and R are completely continuous, the nonzero spectrum of $DR[p_0]$ and the nonzero spectrum of $D\rho[\pi]$ both consist of eigenvalues.

Suppose that $\lambda \in \sigma(DR[p_0])$ with $\lambda \neq 0$. This means that there is some nonzero $y \in A$ such that $DR[p_0]y = \lambda y$. Since $DR[p_0]y = \lambda y \in B$, we in fact have $y \in B$. Therefore

$$D\rho[\pi]DZ[p_0]y = DZ[p_0]DR[p_0]y = DZ[p_0]\lambda y = \lambda DZ[p_0]y.$$

$DZ[p_0]y \neq 0$ since $\ker DZ[p_0]$ is contained in $\ker DR[p_0]$. Thus λ is an eigenvalue of $D\rho[\pi]$ with eigenvector $DZ[p_0]y$.

On the other hand, suppose that $D\rho[\pi]v = \lambda v$, $v \neq 0$, $\lambda \neq 0$. Since $DZ[p_0]$ is surjective, there is some $y \in B$ such that $DZ[p_0]y = v$. Thus we have

$$DZ[p_0]\lambda y = \lambda DZ[p_0]y = \lambda v = D\rho[\pi]v = D\rho[\pi]DZ[p_0]y = DZ[p_0]DR[p_0]y.$$

This means that $DR[p_0]y - \lambda y \in \ker DZ[p_0] \subset \ker DR[p_0]$, and so there is some $u \in \ker DR[p_0]$ such that $DR[p_0]y = \lambda y + u$. Write $\tilde{y} = y + u/\lambda$ and compute:

$$DR[p_0]\tilde{y} = DR[p_0]y = \lambda y + u = \lambda \tilde{y}.$$

This completes the proof. □

We are now ready to state our main theorem, which is really just an assertion of the existence of a certain example. We will give the proof in the next section, applying the above proposition. The essential point of the theorem is that we can obtain instances of (5) with periodic solutions p of oscillation speed 2 such that the spectral radius of $DR[p_0]$, where R is an appropriately defined return map, is greater than one but as close to one as we like. In fact, somewhat more is true: we can achieve spectral radii in a certain range by varying only the delay function d.

Theorem 2.3 (Mildly unstable rapidly oscillating periodic solutions). *Let $\alpha \in (0,1)$ be given. Then there is a particular family of equations* (5), *parameterized by $\gamma \in [0, 1-\alpha]$, of the form*

$$x'(t) = f(x(t - d_\gamma(x(t))))$$

where f is nonincreasing. These equations all satisfy (H1), (H2), and (2); *and for each γ, the following holds: there is a relatively open subset $U \subset \mathcal{D}_0$ and a periodic solution p^γ of oscillation speed 2 such that $p_0^\gamma \in \mathcal{D}_0$ is the fixed point of a return map $R : U \to \mathcal{D}_0$, where $DR[p_0^\gamma]$ has spectral radius*

$$\left(\frac{2}{1+\gamma}\right)^2.$$

In particular, taking $\gamma = 1 - \alpha$ yields a spectral radius of

$$\left(\frac{2}{2-\alpha}\right)^2.$$

Remark 4. In the next section we shall construct p^γ explicitly. By the end of the paper we shall be able to prove, without recourse to the semiconjugacy apparatus described above, a less precise instability result: namely, that there is an open neighborhood W about p_0^γ in $\mathcal{D}_0$ such that, given any $\epsilon > 0$, there are initial conditions $x_0 \in W$ with $\|x_0 - p_0^\gamma\| < \epsilon$ such that $R^m(x_0) \notin W$ for some $m \in \mathbb{N}$. We state such a result precisely below (see Proposition 4). By the end of the paper it will also be easy to recognize, at a heuristic level, the source of the "mildness" of the instability of p^γ when γ is close to 1. We comment in more detail on this below (see Remark 8), but do not formulate a rigorous statement along these lines outside the context of our semiconjugacy framework.

3. **Proof of Theorem 2.3.** In this section, we choose a particular family of state-dependent delay equations and, by dint of fairly explicit calculations and using the apparatus described in Section 2, we show that the family embodies a proof of Theorem 2.3. The family of equations we consider will all have a common nonincreasing feedback function f that is close to a step function in a way we now define.

Definition 3.1. Let $\eta > 0$ be given. We say that $f : \mathbb{R} \to \mathbb{R}$ is *η-steplike* if $f(x) = -\text{sign}(x)$ whenever $|x| \geq \eta$.

We shall take f to be η-steplike for a suitable choice of η.

Remark 5. It has long been recognized that feedback functions that are constant — or almost constant — except on small intervals are very useful in the study of delay equations, since they make explicit computations tractable for specific illustrative examples. The above-mentioned [4] and [12], for example, use feedback functions of this type (with three "nonconstant" intervals instead of one) to exhibit stable rapidly oscillating periodic solutions for certain constant-delay equations; and an early result on stability of a periodic solution of a state-dependent delay equation, with f almost constant outside a small interval, was obtained in [14].

We now describe our family of equations. As in the statement of Theorem 2.3, we first choose and fix $\alpha \in (0, 1)$. We now choose $\eta \in (0, 1/6)$ small enough that

$$\frac{1+\alpha}{\alpha} 7\eta + 3\eta < \frac{1}{2}. \tag{7}$$

Note that this condition implies that

$$\frac{2-\gamma}{1-\gamma} 7\eta + 3\eta < \frac{1}{2} \tag{8}$$

for all $\gamma \in [0, 1-\alpha]$ (since the left-hand side of the above inequality is increasing with respect to $\gamma \in [0, 1-\alpha]$); this is the condition we shall actually use. There is an additional smallness condition that we shall impose on η later (see (13) and (14) below); this condition can be expressed in terms of α also.

For each $\gamma \in [0, 1-\alpha]$ we consider an equation of the form

$$x'(t) = f(x(t - d_\gamma(x(t)))), \tag{9}$$

where f is η-steplike with negative feedback, odd, continuously differentiable with bounded derivative, and nonincreasing. We impose the following hypotheses on d_γ:

- d_γ is even and continuously differentiable;
- $d'_\gamma(x) \le 0$ for all $x \ge 0$, and $|d'_\gamma(x)| \le (\gamma + 1)/2$ for all x;
- $d_\gamma(0) = 1$, and d_γ assumes values in $(0, 1]$;
- $d_\gamma(x) \ge 1 - \gamma x$ for all $x \ge 0$; with $d_\gamma(x) = 1 - \gamma x$ for all $x \in [\eta, 1/2]$.

The function d_γ should be thought of as being essentially the function $x \mapsto 1 - \gamma|x|$, smoothed around the origin and flattened far from the origin.

Observe that (H1) and (H2) hold for all equations (9). (2) holds also: for $(t - d(x(t)))' \ge 1 - (1 + \gamma)/2 = (1 - \gamma)/2$ for any solution x of (9) and any $t > 0$ (recall (6)).

The oddness of f and the evenness of d_γ yield the following lemma.

Lemma 3.2. *Let $x_0, -x_0 \in \mathcal{D}$ have continuations x and y as solutions of* (9), *respectively. Then $x(t) = -y(t)$ for all $t \ge 0$.* □

The following is our main computational lemma. Recall that we are assuming that $\gamma \in [0, 1 - \alpha]$ and that (7) holds.

Lemma 3.3. *Suppose that $x_0 \in \mathcal{D}_0$, and that $x(-d(0)) = x(-1) < -\eta$. Suppose that there is a number $\sigma \in (-1, -1/2]$ such that, on the interval $(-1, \sigma)$, x assumes values in $[-\eta, \eta]$ precisely on an interval of the form $[\zeta - \eta, \zeta + \eta]$, and that on this interval x is given by the formula*

$$x(\zeta + s) = s.$$

Assume also that the following two conditions are satisfied:

$$-1 + \frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta < \zeta - \eta \tag{10}$$

and

$$\sigma > -1 + \frac{2}{1+\gamma}(1 + \zeta) + \frac{2-\gamma}{1-\gamma}\eta + \eta. \tag{11}$$

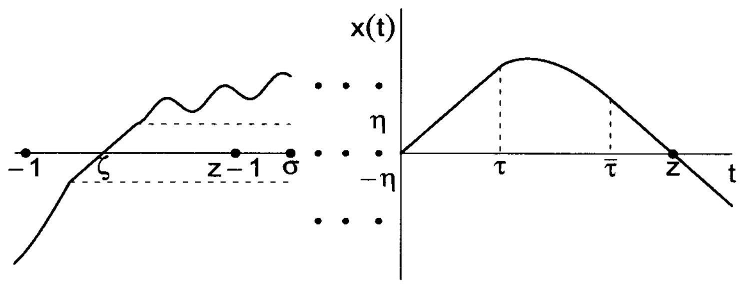
Then the following hold.

- *x has a first positive zero z, and $z - 1 < \sigma$;*
- *$x'(t) = 1$ on $[0, 3\eta]$ and $x'(t) = -1$ on $[z - 3\eta, z]$;*
- *the restriction of x to $[0, z]$ is completely determined by ζ;*
- *the map $\zeta \mapsto z$ is of the form*

$$z =: H(\zeta) = \frac{2 + 2\zeta}{1 + \gamma} + K(\eta),$$

where $K(\eta) \to 0$ as $\eta \to 0$.

See the figure below. (The size of η is exaggerated in the figure for the sake of legibility.)



Remark 6. Suppose that x_0 satisfies the hypotheses of Lemma 3.3, and write y for the continuation of $-x_0$ as a solution of (9). Then, since $y(t) = -x(t)$ for $t \geq 0$ by Lemma 3.2, we see that the first positive zero of y is also given by the formula $z = H(\zeta)$, and that y has constant slope -1 on $[0, 3\eta]$ and constant slope 1 on $[z - 3\eta, z]$. We shall use this fact repeatedly below.

Roughly speaking, (10) describes the minimum distance that ζ can be from -1 while (11) describes the maximum distance that ζ can be from -1 (given σ). Observe that (11) implies in particular that $\zeta + \eta < -1/2$. Condition (7) guarantees that the hypotheses of Lemma 3.3 are not vacuous, as we now explain. For (8) says that

$$-1 + \frac{2-\gamma}{1-\gamma} 7\eta + 3\eta < -\frac{1}{2},$$

which implies that

$$-1 + \frac{2}{1+\gamma}\left(\frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta + \eta\right) + \frac{2-\gamma}{1-\gamma}\eta + \eta < -\frac{1}{2}.$$

Thus there are numbers $\sigma \in (-1, -1/2]$ that satisfy

$$\sigma > -1 + \frac{2}{1+\gamma}\left(\frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta + \eta\right) + \frac{2-\gamma}{1-\gamma}\eta + \eta;$$

and more to the point, given such a σ, there is a range of numbers ζ such that

$$\zeta + 1 > \frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta + \eta$$

(this is (10)) and also such that

$$\sigma > -1 + \frac{2}{1+\gamma}(\zeta + 1) + \frac{2-\gamma}{1-\gamma}\eta + \eta$$

(this is (11)).

Proof. Recall that $(t - d(x(t)))' \geq (1-\gamma)/2$ for all $t > 0$.

Write τ for the unique positive time satisfying $\tau - d(x(\tau)) = \zeta - \eta$. For all $t \in [0, \tau]$, since $x(t - d(x(t))) \leq -\eta$ and f is η-steplike, we have that $x'(t) = 1$ and so $x(t) = t$.

Claim. $\tau \in (3\eta, 1/2)$. Proof of claim: imagine that $\tau \leq 3\eta$. Then since d_γ is nonincreasing on the positive half line we have that

$$\tau - d_\gamma(x(\tau)) \leq 3\eta - d_\gamma(3\eta) = -1 + (1+\gamma)3\eta < -1 + \frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta < \zeta - \eta,$$

a contradiction (the last inequality is (10)). Thus we see that $\tau > 3\eta$. On the other hand, if $\tau \geq 1/2$ we would have

$$\frac{1}{2} - d_\gamma\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{\gamma}{2} \leq \zeta - \eta,$$

contradicting (11) (recall that (11) implies that $\zeta + \eta < \sigma \leq -1/2$). This proves the claim. Since $\tau \in (3\eta, 1/2)$, we can use our explicit formula for d_γ on $[\eta, 1/2]$ to obtain

$$\tau - 1 + \gamma\tau = \zeta - \eta \implies \tau = \frac{1 + \zeta - \eta}{1 + \gamma}.$$

Let us write

$$\bar{\tau} = \min\{t > \tau \ : \ x(t) = 3\eta \text{ or } x(t) = 1/2 \text{ or } t - d(x(t)) = \zeta + \eta\},$$

and let us write $y(t) = t - d(x(t)) - \zeta$.

Here is where we use our elaborate hypothesis on the form of x on $[\zeta - \eta, \zeta + \eta]$. For all $t \in [\tau, \bar{\tau}]$, we have

$$x(t - d(x(t))) = x(y(t) + \zeta) = y(t) \quad\text{and}\quad y'(t) = 1 - d_\gamma'(x(t))x'(t) = 1 + \gamma x'(t).$$

Therefore, for $t \in [\tau, \bar{\tau}]$, $x(t)$ and $y(t)$ solve the following ODE:

$$\begin{aligned} &x'(t) = f(y(t)); \\ &y'(t) = 1 + \gamma f(y(t)); \\ &x(\tau) = \tau, \quad y(\tau) = -\eta. \end{aligned}$$

Let us write $(X(t), Y(t))$ for the unique solution of this ODE on $[\tau, \infty)$. Let us further write $c > \tau$ for the unique time that $Y(c) = 0$, and $T > c$ for the unique time that $Y(T) = \eta$. For $t \in [\tau, \bar{\tau}]$, we of course have $(x(t), y(t)) = (X(t), Y(t))$. We wish to show that $\bar{\tau} = T$ — that is, that $\bar{\tau}$ is characterized by the condition that $y(\bar{\tau}) = \eta$ (as opposed to $x(\bar{\tau}) = 1/2$ or $x(\bar{\tau}) = 3\eta$). To do this, it suffices to show that $X(t) \in (3\eta, 1/2)$ for all $t \in [\tau, T]$, and we now do so.

Note that X is increasing on $[\tau, c]$ and decreasing on $[c, T]$. Since $Y'(t) \geq 1$ on $[\tau, c]$ and $Y'(t) \geq (1 - \gamma)/2$ on $[c, T]$, we have the bounds $c - \tau \leq \eta$ and $T - c \leq 2\eta/(1 - \gamma)$. The bound $|X'(t)| \leq 1$ now yields $X(c) \leq \tau + \eta$ and $X(T) \geq \tau - 2\eta/(1 - \gamma)$. Using our assumption that $\zeta < -1/2$, the fact that $\eta \in (0, 1/6)$, and our formula for τ we obtain

$$\begin{aligned} & \zeta < -\frac{1}{2} + \gamma\left(\frac{1}{2} - \eta\right) \\ \implies\quad & 1 + \zeta - \eta + \eta + \gamma\eta < \frac{1}{2} + \frac{\gamma}{2} \\ \implies\quad & \frac{1 + \zeta - \eta}{1 + \gamma} + \eta < \frac{1}{2} \\ \implies\quad & \tau + \eta < \frac{1}{2}. \end{aligned}$$

Thus $X(c) < 1/2$. On the other hand, applying (10) we get

$$\begin{aligned} & \zeta - \eta > -1 + \frac{(1+\gamma)(2-\gamma)}{1-\gamma} 3\eta \\ \implies \quad & \zeta - \eta > -1 + \left[\frac{1+\gamma}{1-\gamma} + \frac{1-\gamma^2}{1-\gamma}\right] 3\eta \\ \implies \quad & \zeta - \eta > -1 + \left[\frac{2(1+\gamma)}{1-\gamma} + (1+\gamma)3\right] \eta \\ \implies \quad & 1 + \zeta - \eta - \frac{2(1+\gamma)}{1-\gamma}\eta > (1+\gamma)3\eta \\ \implies \quad & \tau - \frac{2\eta}{1-\gamma} > 3\eta. \end{aligned}$$

Thus we have that $X(T) > 3\eta$. We conclude that $X(t) \in (3\eta, 1/2)$ for all $t \in (\tau, T)$, that $T = \bar{\tau}$, and that $\bar{\tau}$ is characterized by the equality $y(\bar{\tau}) = \eta$.

Now, observe that the restriction of y to $[\tau, \bar{\tau}]$ does not depend on ζ — or, indeed, on anything about the initial condition x_0 except that it satisfy the hypotheses of Lemma 3.3: for on $[\tau, \bar{\tau}]$, y is actually the solution of a one-dimensional ODE with fixed initial condition $y(\tau) = -\eta$. $x(\bar{\tau}) - \tau$, likewise — indeed, the restriction of $x(t) - \tau$ to $[\tau, \bar{\tau}]$ — does not depend on ζ either. We conclude that $\kappa_1 := \bar{\tau} - \tau$ and $\kappa_2 := x(\bar{\tau}) - x(\tau)$ are constant — with respect to ζ — and satisfy the following bounds (since we actually have that $x(t) \in (3\eta, 1/2)$ and hence that $Y'(t) \geq (1-\gamma)$ on $[c, \bar{\tau}]$):

$$\begin{aligned} 0 \leq \kappa_1 \leq \eta + \frac{\eta}{1-\gamma} = \eta\frac{2-\gamma}{1-\gamma}; \\ -\frac{\eta}{1-\gamma} \leq \kappa_2 \leq \eta. \end{aligned}$$

Observe that $\kappa_1, \kappa_2 \to 0$ (uniformly as γ ranges over $[0, 1-\alpha]$) as $\eta \to 0$.

Let us write t_* for the unique time that $t_* - d(x(t_*)) = \sigma$. Observe that $x'(t) = -1$ for all $t \in [\bar{\tau}, t_*]$ (since $x(t - d(x(t))) \geq \eta$ for all such t). Let us imagine that $x(t) \geq 0$ for all $t \in [\bar{\tau}, t_*]$; we shall derive a contradiction to (11). Observe that

$$x(t_*) = x(\bar{\tau}) - (t_* - \bar{\tau}) = 2\tau + \kappa_1 + \kappa_2 - t_*.$$

If $x(t_*) \geq 0$, then since $d_\gamma(x(t_*)) \geq 1 - \gamma x(t_*)$ we have

$$\begin{aligned} \sigma = t_* - d_\gamma(x(t_*)) \quad & \leq \quad t_* - 1 + \gamma x(t_*) \\ & \leq \quad (1-\gamma)t_* - 1 + 2\gamma\tau + \gamma(\kappa_1 + \kappa_2). \end{aligned}$$

Still assuming that $x(t_*) \geq 0$, we must have

$$t_* \leq \bar{\tau} + x(\bar{\tau}) = 2\tau + \kappa_1 + \kappa_2,$$

whence

$$\sigma \leq -1 + 2\tau + \kappa_1 + \kappa_2.$$

But our condition on σ is that

$$\begin{aligned} & \sigma > -1 + \frac{2}{1+\gamma}(1+\zeta) + \frac{2-\gamma}{1-\gamma}\eta + \eta \\ \implies \quad & \sigma > -1 + \frac{2(1+\zeta-\eta)}{1+\gamma} + \kappa_2 + \kappa_1 \\ \implies \quad & \sigma > -1 + 2\tau + \kappa_2 + \kappa_1; \end{aligned}$$

a contradiction.

We conclude that the first positive zero z of x occurs before time t_*. We compute $z - \bar{\tau} = x(\bar{\tau}) = \tau + \kappa_2$ and we have the formula

$$z = \tau + \kappa_1 + (\tau + \kappa_2) = 2\tau + \kappa_1 + \kappa_2 = \frac{2 + 2\zeta}{1+\gamma} + K(\eta) =: H(\zeta),$$

where $K(\eta) = \kappa_1 + \kappa_2 - \frac{2\eta}{1+\gamma}$. It is clear that the restriction of x to $[0, z]$ is determined by ζ. This completes the proof. □

We shall henceforth write H for the function $\zeta \mapsto z$ described above.

Direct computation establishes the following lemma, which will be instrumental in the construction of our periodic solution of (9).

Lemma 3.4. *The equation $H(-2z) = z$ has a unique solution*

$$\bar{z} = \frac{2}{5+\gamma} + \frac{1+\gamma}{5+\gamma}K(\eta). \quad \square \tag{12}$$

We now impose our second size condition on η. Since $K(\eta) \to 0$ as $\eta \to 0$, for any $\gamma \in [0, 1-\alpha]$ and all η small enough, $\bar{z}$ is approximately equal to $2/(5+\gamma)$. This observation along with some computation shows that we can choose η small enough, relative to α, such that (for any f satisfying our hypotheses and any $\gamma \in [0, 1-\alpha]$) the following hold.

$$-3\bar{z} + 6\eta < -1 < -2\bar{z} - 6\eta \tag{13}$$

and

$$(10) \text{ and } (11) \text{ hold for any } \zeta \in [-2\bar{z} - \eta, -2\bar{z} + \eta],\ \sigma = -1/2. \tag{14}$$

We assume henceforth that η is so small: that is, (7), (13) and (14) hold given any f satisfying our hypotheses and any $\gamma \in [0, 1-\alpha]$. Observe in particular that

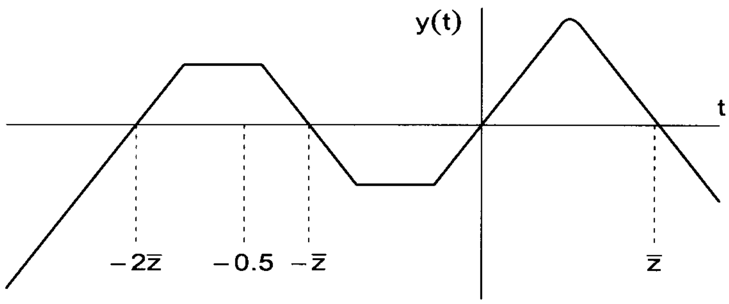
$$\bar{z} > 12\eta \quad \text{and} \quad -2\bar{z} + 3\eta < -1/2 < -\bar{z} - 3\eta.$$

(The first inequality on the right comes from multiplying (13) through by 2/3.)

Lemma 3.3 now allows us to give an explicit description of a periodic solution of (9) with oscillation speed 2. Still with $\alpha \in (0,1)$ fixed, choose and fix $\gamma \in [0, 1-\alpha]$ and assume that our standing hypotheses on (9) hold — in particular, that η satisfies (7), (13) and (14). Let $\bar{z}$ be as described above. Now choose $y_0 \in \mathcal{D}_0$ satisfying the following conditions:

- $y_0(s) = s$ for all $s \in [-\eta, 0]$;
- $y_0(s) < -\eta$ for all $s \in (-\bar{z} + \eta, -\eta)$;
- $y_0(s) = -\bar{z} - s$ for all $s \in [-\bar{z} - \eta, -\bar{z} + \eta]$;
- $y_0(s) > \eta$ for all $s \in (-2\bar{z} + \eta, -\bar{z} - \eta)$;
- $y_0(s) = s + 2\bar{z}$ for all $s \in [-2\bar{z} - \eta, -2\bar{z} + \eta]$;
- $y_0(s) < \eta$ for all $s \in [-1, -2\bar{z} - \eta)$.

Let us write y for the continuation of y_0 as a solution of (9), and let us write $\xi_1 < \xi_2 < \xi_3 < \xi_4$ for the first four positive zeros of y. Note that y_0 satisfies the hypotheses of Lemma 3.3, with $-2\bar{z}$ playing the role of ζ and $\sigma = -1/2$. Thus $y'(t) = 1$ on $[0, \eta]$, $y'(t) = -1$ on $[\xi_1 - \eta, \xi_1]$, and $\xi_1 = H(-2\bar{z}) = \bar{z}$. Moreover, $\xi_1 - d(0) = \bar{z} - 1 \in (-2\bar{z} + 6\eta, -\bar{z} - 6\eta)$ by (13), so $y(\bar{z} - 1) > \eta$. Thus $-y_{\bar{z}}$ satisfies exactly the same conditions that we listed for y_0 above, and so in particular satisfies the the hypotheses of Lemma 3.3 too, with the role of ζ played by $-\bar{z} - \bar{z} = -2\bar{z}$. See the figure below.



Using the symmetry result of Lemma 3.2, we conclude that $\xi_2 = 2\bar{z}$ and that $y_{2\bar{z}}$ satisfies the conditions given for y_0. Continuing this way shows that $y_{4\bar{z}}$ also has zeros $-2\bar{z}$ and $-\bar{z}$ on $(-1,0)$ and satisfies the conditions given for y_0, and hence the hypotheses of Lemma 3.3. Therefore, by Lemma 3.3 we see that the restriction of y to $[4\bar{z}, 5\bar{z}]$ will be equal to (a translate of) the restriction of y to $[0, \bar{z}]$, and so on. Continuing this reasoning shows that

$$y_{4\bar{z}} = y_{6\bar{z}} = y_{8\bar{z}} = \cdots$$

— that is, that $y_{4\bar{z}}$ is the fixed point of a return map R that "advances solutions by four zeros." In particular, we see that (9) has a periodic solution p^γ of constant oscillation speed 2 with $p_0^\gamma = y_{4\bar{z}}$.

(The necessity of "advancing by four zeros" comes from the fact that 4 is the smallest positive even integer k such that $k\bar{z} > 1$. This condition is required to make the map R completely continuous — recall Section 2. We also emphasize that $y_{2\bar{z}}$ is *not* necessarily a segment of a periodic solution: the restriction of $y_{2\bar{z}}$ to $[-1, -2\bar{z}]$ is just the restriction to $[2\bar{z}-1, 0]$ of the initial condition y_0, and there is no reason to expect this restriction to be equal to the restriction of $y_{4\bar{z}}$ to $[-1, -2\bar{z}]$.)

Remark 7. Essentially the same approach that we have used above can be used to show that, given any positive even n, (9) has a periodic solution of oscillation speed n for all η small enough. The spacing $\tilde{z}_n$ between the zeros will be the solution of the equation $H(-n\tilde{z}_n) = \tilde{z}_n$, and the periodic solution can be obtained (provided η is small enough) by repeatedly applying Lemma 3.3, with $\sigma \in (-n\tilde{z}_n, -(n-1)\tilde{z}_n)$, to an initial condition analogous to y_0 but with zeros $\tilde{z}_n$ units apart. We do not present the details here.

For notational simplicity, we write $p := p^\gamma$ henceforth. We shall spend most of the rest of the paper defining the return map R carefully, putting ourselves into the semiconjugacy framework of Section 2, and showing that $DR[p_0]$ has spectral radius $(2/(1+\gamma))^2$; this will complete the proof of Theorem 2.3. The ideas will all be familiar to readers accustomed to dealing with feedback functions that are similar to step functions.

We collect some facts about p. These follow from the periodicity of p, Lemma 3.2, our conditions on $\bar{z}$, and the proof of Lemma 3.3 (particularly that, in the notation of that lemma, $3\eta < \tau < \bar{\tau} < z - 3\eta$; and so $p(t)$ has slope ± 1 on intervals of radius strictly greater than 3η about each of its zeros, and $|p(t)| > 3\eta$ away from those intervals).

Lemma 3.5. *p has constant oscillation speed 2, and the following hold.*

- *The zeros of p are precisely the numbers $k\bar{z}$, $k \in \mathbb{Z}$.*
- *p has slope $(-1)^k$ on the intervals $[k\bar{z} - 3\eta, k\bar{z} + 3\eta]$, and these intervals of radius 3η about the points $k\bar{z}$ are precisely the intervals for which $|p(t)| \leq 3\eta$.*
- *$-3\bar{z} + 3\eta < -1$, so $|p_0(s)| \leq 3\eta$ precisely for s on the subset $[-2\bar{z} - 3\eta, -2\bar{z} + 3\eta] \cup [-\bar{z} - 3\eta, -\bar{z} + 3\eta] \cup [-3\eta, 0]$.*
- *$p(t + \bar{z}) = -p(t)$ for all t.*
- *There is a positive number $\beta > 0$ such that, for all $k \in \mathbb{Z}$, $|p(t-d(p(t)))| \geq \eta+\beta$ for all $t \in [k\bar{z} - 3\eta, k\bar{z} + 3\eta]$.* $\square$

Notation. We shall henceforth write $X = \mathcal{D}_0$, and take $U \subset X$ to be a relatively open ball in X about p_0 of radius δ.

We now define the somewhat more intricate set Y.

Definition 3.6. Y is the subset of members y_0 of X satisfying the following additional requirements:

- $\|y_0 - p_0\| < \eta$;
- $y_0'(s) = 1$ on $[-2\eta, 0]$ and on $[-2\bar{z} - 2\eta, -2\bar{z} + 2\eta]$;
- $y_0'(s) = -1$ on $[-\bar{z} - 2\eta, -\bar{z} + 2\eta]$.

We collect some immediate consequences of Definition 3.6.

Lemma 3.7. *Every $y_0 \in Y$ has precisely two zeros $\zeta_{-2} < \zeta_{-1}$ on $(-1, 0)$, with*

$$|\zeta_{-2} - (-2\bar{z})| < \eta \quad \text{and} \quad |\zeta_{-1} - (-\bar{z})| < \eta.$$

Furthermore,

$$\begin{aligned} |y_0(s)| \leq \eta \quad \Longrightarrow \quad & s \in [\zeta_{-2} - \eta, \zeta_{-2} + \eta] \cup [\zeta_1 - \eta, \zeta_{-1} + \eta] \cup [-\eta, 0] \\ \subset \quad & [-2\bar{z} - 2\eta, -2\bar{z} + 2\eta] \cup [-\bar{z} - 2\eta, -\bar{z} + 2\eta] \cup [-2\eta, 0]. \end{aligned}$$

Since $-\bar{z} - 2\eta > -\bar{z} - 3\eta > -1/2$ (this follows from (13)), we see in particular that, on $[-1, -1/2]$, $|y_0(s)| \leq \eta$ precisely on the interval $[\zeta_{-2} - \eta, \zeta_{-2} + \eta]$, where y_0 has constant slope 1. Using the notation introduced in Lemma 3.7, (14) now yields

Lemma 3.8. *Any point $y_0 \in Y$ satisfies the hypotheses of Lemma 3.3, with ζ_{-2} in the role of ζ and $\sigma = 1/2$.*

X and Y are of course subsets of the Banach space $C^1[-1, 0]$; we now show that X and Y have the local submanifold structure that we need to apply the semiconjugacy apparatus described in Section 2. In fact, near p_0, the submanifold structures of X and Y are particularly simple: X and Y are locally affine, as we now explain. Since $p(-d(0)) = p(-1) \leq -\eta - \beta$, for all $x_0 \in X$ sufficiently close to p_0 (remember that X has the C^1 metric) we have that $x_0'(0) = 1$. Thus any sufficiently small open set in X about p_0 is the intersection of an open set in C^1 and the affine space $p_0 + A$, where

$$A = \{v \in C^1 : \ v'(0) = 0 \}.$$

Similarly, since the members of Y have prescribed slope on certain intervals, any sufficiently small open set in Y about p_0 is the intersection of an open set in C^1 and the affine space $p_0 + B$, where

$$B = \{ v \in C^1 \ : \ v'(s) = 0 \text{ for all } s \in [-2\bar{z}-2\eta, -2\bar{z}+2\eta] \cup [-\bar{z}-2\eta, -\bar{z}+2\eta] \cup [-2\eta, 0] \}.$$

Thus X and Y have the structure required for the semiconjugacy framework of Section 2. (Notice that the tangent spaces to X and Y at p_0 are $T_{p_0}X = A$ and

$T_{p_0}Y = B$, respectively.) In the notation introduced in Section 2, we shall take $V = \mathbb{R}^2$ below, equipped with the usual Euclidean norm. To apply the semiconjugacy framework, then, we have the required spaces; we need to define the maps R, Z and ρ, and to prove properties (I) – (IV).

We begin with the map $Z : Y \to \mathbb{R}^2$, which we define by the formula (again using the notation introduced in Lemma 3.7, which we shall use for the rest of the paper)

$$Z(y_0) = (-\zeta_{-1}, -\zeta_{-2}).$$

We extend Z to $-Y$ by defining $Z(y_0) := Z(-y_0)$ for all $y_0 \in -Y$.

$Z(y_0)$ is given by the following formula: for $y_0 \in Y$,

$$Z(y_0) = (\bar{z} - y_0(-\bar{z}), 2\bar{z} + y_0(-2\bar{z})).$$

Z is affine, and so it is clear that Z is continuous, open, and continuously differentiable with $DZ[p_0] : T_{p_0}Y \to \mathbb{R}^2$ surjective.

We now define the map R, and give its important properties. R is a return map, defined near p_0, that "advances solutions by four zeros". Any fixed point of R is a segment of a periodic solution. (Recall that we are viewing f, η, γ, p, and Y as fixed.)

Lemma 3.9. *There is a $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0]$, the following hold.*

Write U for the relatively open ball of radius δ in X with center p_0. Given $y_0 \in U$ with continuation y as a solution of (9),

- *The first four positive zeros $z_1 < z_2 < z_3 < z_4$ of y are defined and proper;*
- *The return map $R : U \to X$ defined by*

  $$R(y_0) = y_{z_4}$$

  is continuously differentiable and completely continuous;
- *$R(U) \subset Y$;*
- *If in fact $y_0 \in U \cap Y$, then*

  $$y_0, \ -y_{z_1}, \ y_{z_2}, \ -y_{z_3}, \text{ and } y_{z_4}$$

  all satisfy the hypotheses of Lemma 3.3, with $\sigma = -1/2$ and the role of ζ played, respectively, by

  $$\zeta_{-2}, \ \zeta_{-1} - z_1, \ -z_2, \ z_1 - z_3, \text{ and } z_2 - z_4.$$

Proof. For notational simplicity we write $d_\gamma = d$ henceforth.

Suppose that $x(t)$ and $y(t)$ are two solutions of (9). Write b for an upper bound on $|f'|$, D for an upper bound on $|d'|$, and M such that $|y'(t)| \le M$ for all $t \in [-1, \infty)$. (We may take $D = (1+\gamma)/2$, and $M = \max\{\|y_0\|, 1\}$, since $|y'(t)| \le 1$ for all $t \ge 0$.) Write $\|\cdot\|_0$ for the sup norm. For $t \ge 0$, we then have the elementary bounds

$$\begin{aligned}
&|x'(t) - y'(t)| \\
\le&|f(x(t - d(x(t)))) - f(y(t - d(y(t))))| \\
\le&|f(x(t - d(x(t)))) - f(y(t - d(x(t))))| + |f(y(t - d(x(t)))) - f(y(t - d(y(t))))| \\
\le&b\left[|x(t - d(x(t))) - y(t - d(x(t)))| + |y(t - d(x(t))) - y(t - d(y(t)))|\right] \\
\le&b\|x_t - y_t\|_0 + bMD\|x_t - y_t\|_0 = b(1 + MD)\|x_t - y_t\|_0.
\end{aligned}$$

This Lipschitz bound now allows us to establish, by familiar arguments, the following: that given any $\epsilon > 0$, we may choose δ such that $\|y_0 - p_0\| \le \delta$ implies that $\|F(t, y_0) - F(t, p_0)\| \le \epsilon$ for all $t \in [0, 4\bar{z} + 3\eta]$.

Since f and d are smooth, we have the estimate $|x''(t)| \le b(D+1)$ for all $t \ge 0$. This implies in particular that $p'(t)$ has a Lipschitz constant ℓ. We now choose the ϵ in the last paragraph to satisfy $\epsilon < \eta/(2+\ell)$.

In addition, using the estimate (related to the estimate above)

$$|x(t-d(x(t))) - y(t-d(y(t)))| \le (1+MD)\|x_t - y_t\|_0$$

and recalling from Lemma 3.5 that $|p(t-d(p(t))| \ge \eta + \beta$ whenever t is within 3η of a zero of p, we also choose ϵ so small that $|y(t-d(y(t))) - p(t-d(p(t)))| \le \beta$ — and so $y'(t) = p'(t)$ — for all t in

$$[0, 3\eta] \cup [\bar{z} - 3\eta, \bar{z} + 3\eta] \cup [2\bar{z} - 3\eta, 2\bar{z} + 3\eta] \cup [3\bar{z} - 3\eta, 3\bar{z} + 3\eta] \cup [4\bar{z} - 3\eta, 4\bar{z} + 3\eta].$$

On each of these intervals, we also of course have $|y(t) - p(t)| < \epsilon < \eta/(2+\ell)$. Writing $z_1 < z_2 < z_3 < z_4$ for the first four positive zeros of y, we see that each of these zeros is well-defined and proper, with $|z_k - k\bar{z}| < \eta/(2+\ell)$ for all $k \in \{1,2,3,4\}$. We also have that $\|y_{k\bar{z}} - p_{k\bar{z}}\| \le \eta/(2+\ell)$ for all $k \in \{1,2,3,4\}$. Recalling that the Lipschitz constant of p is 1 and the Lipschitz constant of p' is ℓ, we have the C^1-norm estimate

$$\|y_{z_k} - p_{k\bar{z}}\| \le \|y_{k\bar{z}} - p_{k\bar{z}}\| + |z_k - k\bar{z}|(\ell+1) < (1+1+\ell)\eta/(2+\ell) = \eta$$

for all $k \in \{1,2,3,4\}$.

As in the statement of the lemma, we now define the map $R : U \to X$ by $R(y_0) = y_{z_4}$. We have just shown that $\|R(y_0) - p_0\| < \eta$.

We now show that $R(y_0) \subset Y$. Since $y(t)$ has constant slope 1 on $[2\bar{z}-3\eta, 2\bar{z}+3\eta]$, constant slope -1 on $[3\bar{z} - 3\eta, 3\bar{z} + 3\eta]$, and constant slope 1 on $[4\bar{z} - 3\eta, 4\bar{z} + 4\eta]$, translating by a distance of magnitude $|z_4 - 4\bar{z}| < \eta$ we get that y_{z_4} has constant slope 1 on $[-2\bar{z} - 2\eta, -2\bar{z} + 2\eta]$, constant slope -1 on $[-\bar{z} - 2\eta, -\bar{z} + 2\eta]$, and constant slope 1 on $[-2\eta, 0]$. Thus (since we already know that $\|R(y_0) - p_0\| < \eta$) we have that $R(y_0) \in Y$, as desired.

The proof of the last part of the lemma is similar. That y_0 and $R(y_0) = y_{z_4}$ satisfy the hypotheses of Lemma 3.3 is of course just Lemma 3.8. For $k \in \{1,2,3\}$, a translation argument just like in the last paragraph shows that

- y_{z_k} has precisely two zeros $\hat{\zeta}_{-2} < \hat{\zeta}_{-1}$ on $(-1, 0)$;
- y_{z_k} has constant slope ± 1 on each of the intervals $[\hat{\zeta}_{-2} - \eta, \hat{\zeta}_{-2} + \eta]$, $[\hat{\zeta}_{-1} - \eta, \hat{\zeta}_{-1} + \eta]$, and $[-\eta, 0]$;
- The above listed intervals are exactly the subintervals of $[-1, 0]$ for which $|y_{z_k}(s)| \le \eta$;
- $|\hat{\zeta}_{-2} - (-2\bar{z})| < \eta$ and $\hat{\zeta}_{-1} - \eta > -1/2$.

That each of these y_{z_k} (or its negative) satisfies the hypotheses of Lemma 3.3 now follows from (14). The formulas for the quantities playing the role of ζ are now obvious.

The smoothness and complete continuity of the map $y_0 \mapsto y_{z_4} = R(y_0)$ follow from standard arguments, as asserted in Section 2. ☐

We shall henceforth assume that δ, U and R are as in Lemma 3.9.

In the notation of the general semiconjugacy framework described in Section 2, we have established hypothesis (I) and parts of (II) and (III). We now turn to computing the semiconjugating map ρ.

Writing (u, v) for the general point of $\mathbb{R}^2$, let us introduce the two-dimensional affine map

$$\Psi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} H(-v) \\ u + H(-v) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{1+\gamma} \\ 1 & -\frac{2}{1+\gamma} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \frac{2}{1+\gamma} + K(\eta) \\ \frac{2}{1+\gamma} + K(\eta) \end{pmatrix}.$$

Ψ has the unique fixed point $(\bar{z}, 2\bar{z})$.

Proposition 3. *Let $y_0 \in U \cap Y$, with δ as in Lemma 3.9. Then*

$$Z(R(y_0)) = \rho(Z(y_0)),$$

where $\rho = \Psi^4$.

Proof. By this point, the proof is perhaps obvious. Since y_0 and y_{z_k}, $k \in \{1, 2, 3, 4\}$ (or their negatives) all satisfy the hypotheses of Lemma 3.3 with the role of ζ as described in Lemma 3.9, we compute

$$z_1 = H(\zeta_{-2});\ z_2 - z_1 = H(\zeta_{-1} - z_1);\ z_3 - z_2 = H(-z_2);\ z_4 - z_3 = H(z_1 - z_3).$$

Computation now shows that $Z(R(y_0)) = (z_4 - z_3, z_4 - z_2)$ is equal to $\rho(Z(y_0))$. □

We have verified hypothesis (IV) of the framework described in Section 2.

Suppose that $y_0 \in U \cap Y$. Lemma 3.3 (together with Lemma 3.9) tells us that ζ_{-2} completely determines the restriction of y to $[0, z_1]$, that $\zeta_{-1} - z_1$ completely determines the restriction of y to $[z_1, z_2]$, and so on. Since $z_1 - \zeta_{-1}$, though, is just the second coordinate of $\Psi(Z(y_0))$, we see that $Z(y_0)$ completely determines the restriction of y to $[0, z_2]$. Continuing this reasoning shows us that $Z(y_0)$ in fact determines the entire restriction of y to $[0, z_4]$. Since $z_4 > 1$, we see that $Z(y_0)$ determines $R(y_0)$ — more precisely, if $x_0, y_0 \in U \cap Y$ and $Z(y_0) = Z(x_0)$, then $R(y_0) = R(x_0)$. This completes the verification of hypothesis (II) in our semiconjugacy framework.

Since Z is affine, to say that $v \in T_{p_0}Y$ is in the kernel of $DZ[p_0]$ is to say that $Z(p_0 + hv) = Z(p_0)$ for all h real with $|h|$ sufficiently small. It follows from (II) that $R(p_0 + hv) = R(p_0)$ for all such h, and so $v \in \ker DR[p_0]$ too. This completes the verification of hypothesis (III) from Section 2. All necessary hypotheses for the framework have now been established.

Direct computation yields that the spectral radius of $D\rho[Z(p_0)]$ is $\sqrt{2/(1+\gamma)}^{\,4} = (2/(1+\gamma))^2$. We now apply Proposition 2 to conclude that $DR[p_0]$ has spectral radius $(2/(1+\gamma))^2$ also. This completes the proof of Theorem 2.3.

Even without the semiconjugacy framework, we can see that p is unstable in a sense that we now describe (this is the standard "nonlinear" definition of instability of a fixed point).

Proposition 4. *Given any $\epsilon > 0$, there is a point $x_0 \in U$ with $\|x_0 - p_0\| < \epsilon$ such that $R^m(x_0) \notin U$ for some m.*

Proof. There are points $x_0 \in U \cap Y$ with $Z(x_0) \neq Z(p_0)$ and $\|x_0 - p_0\| < \epsilon$. The sequence $\rho^m(Z(x_0))$ is unbounded in $\mathbb{R}^2$, for ρ is affine with spectral radius greater than 1.

For any $N \in \mathbb{N}$ such that $R^k(x_0) \in U$ for all $k \in \{0, \ldots, N-1\}$, by Lemma 3.9 we in fact have that $R^k(x_0) \in U \cap Y$ for all such k. Repeated application of Proposition 3 now yields that $\rho^N(Z(x_0)) = Z(R^N(x_0))$. If we imagine that $R^N(x_0) \in U$ for all $N \in \mathbb{N}$, then, we see that $Z(R^N(x_0))$ must be unbounded; but very crude estimates show that any point in $Z(Y)$ has Euclidean norm no more than $\sqrt{1^2 + (1/2)^2} = \sqrt{5}/2$. □

Remark 8. For γ close to 1, the linear part of the affine map Ψ is close to

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

$A^3 v = v$ for all $v \in \mathbb{R}^2$. Thus, for γ close to 1, the orbit of any point near $Z(p_0) = (\bar{z}, 2\bar{z})$ under Ψ will be close to a period-three orbit, slowly spiraling away from $Z(p_0)$. Thus, for γ close to 1 and $x_0 \in U \cap Y$ close to p_0, the continuation x of x_0 as a solution of (9) will remain (up to translation) close to p for a large interval of positive time, with the spacing between the zeros forming an approximate 3-periodic sequence (this phenomenon is readily observed in numerical approximations). Thus we see, at a heuristic level, how taking γ close to 1 makes the instability of p "mild."

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E-mail address: **bkennedy@gettysburg.edu**