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Chutes and Ladders with Large Spinners

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Chutes and Ladders with Large Spinners

Abstract

We prove a conjecture from a 2011 College Mathematics Journal article addressing the expected number of turns in a Chutes and Ladders game when the spinner range is close to the length of the board. While the original paper approached the question using linear algebra and the theory of Markov processes, our main method uses combinatorics and recursion.

Keywords

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Darcie Connors and Darren Glass



Darcie Connors (connda02@alumni.gettysburg.edu) received her BA in mathematics at Gettysburg College in 2014, where she explored this paper's topic for her senior capstone. Her interest began with a love of board games but was further fueled by the questions posed by the authors of a 2011 paper from this JOURNAL. She has just started teaching mathematics at a middle school in Boston.



Darren Glass (dglass@gettysburg.edu) received his BA in mathematics and economics at Rice University and his PhD at the University of Pennsylvania, where he worked on questions in arithmetic geometry and Galois theory. Since joining the mathematics department at Gettysburg College in 2005, his research interests have expanded to include graph theory, cryptography, and whatever interesting questions students want to talk about. This paper is evidence of the latter.

The game of *Chutes and Ladders* has been a popular and widely played game since its origin in Asia as a game of “moral instruction” [8]. It is also commonly referred to as *Snakes and Ladders*, the name used in England since the late 1800s. Though the game has changed over time, the main ideas have remained the same, slowly evolving to the popular children’s game still played in the United States today.

The game as it is manufactured by Hasbro is played on a board made up of 100 squares. It is a simple race game: Players move their game pieces from square to square along the track toward the last square and movement involves no strategy or choice. On each turn, players spin a spinner that moves them between one and six squares with equal probability. The only deviations from this simple game are the eponymous chutes and ladders, which are scattered throughout the board, each connecting two squares. If a player’s spin moves her to the square marking the lower end of a ladder, then she moves up the ladder to the higher square, getting closer to the goal of reaching the last square of the game. Conversely, if her spin deposits her at the top of a chute, then she must slide down to the square marking the bottom. Play continues in this way until one player’s game piece reaches the last square and wins. It is important to note that the winner must land *exactly* on the last square. If she spins a number that is greater than the number of squares left between her current position and the end of the game, then she remains on her current square until her next turn.

Despite its simplicity and popularity among younger children, this game has prompted some interesting mathematical questions, including those explored in [1, 2, 4, 6, 7]. Leslie A. Cheteyan, Stewart Hengeveld, and Michael A. Jones [3] investigate how changing the range of the spinner might change the expected length of

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a game. For example, would a game go faster if instead of the traditional spinner one played with a spinner of range of 20, where each number 1 through 20 is as equally likely to be spun? Or a range of 100, the length of the board game? One might guess that using a spinner with a larger range would decrease the expected length, as the expected value of a spin is greater so the player has the possibility of moving more quickly toward the end of the game by spinning higher numbers. Recall, however, that a game piece must land on the last square exactly, so a larger spinner range may result in more missed turns at the end when the player has a smaller probability of being able to move at all. They investigate how these two opposing pieces of intuition interact with each other to try to determine what spinner range leads to the shortest game.

The authors of [3] model *Chutes and Ladders* as a Markov chain, with each square representing a different state. By carefully writing down the transition matrix and its fundamental form, they compute the expected game length for spinners of various ranges and their paper gives results for several simplified versions of the game as well as the officially manufactured version. They make the observation that for boards with n squares it appears that the expected number of turns is the same if a player uses a spinner of range n or one of range $n - 1$. They pose this as an open question, and in the remainder of the note we will prove that their observation holds true.

Theorem 1. *For any Chutes and Ladders board with n squares, the expected number of turns until a game piece starting at square 0 lands exactly on square n is the same if one uses a spinner of range n or $n - 1$.*

One can make a good deal of headway into this problem by using the Markov chain techniques of [3]. In fact, this was a significant part of the work done by the first author of this article in her senior thesis at Gettysburg College. However, here we take a more direct combinatorial approach. To illustrate the general idea, we consider in the next section a board without chutes and ladders. Subsequent sections develop the general case. In each of our cases, E denotes the expected number of moves it will take to get from square 0 to square n using a spinner of range n and F denotes the expected number of turns using a spinner of range $n - 1$; our goal is to prove that $E = F$.

No chutes or ladders

To begin considering our question, we simplify it as much as possible. The chutes and ladders set this game apart from other simple race games and make play more interesting, but they also complicate the analysis, so in this section we consider a game board with no chutes or ladders. In order to do so, we will make repeated use of the following theorem, a basic result in probability theory.

Theorem 2. *Consider a process in which each trial is independent and will succeed with probability p and fail with probability $1 - p$. If we keep repeating the process and stop after our first success, then the expected number of trials is $\frac{1}{p}$.*

See [5] for one proof. This situation is known in the literature as a *Bernoulli process* and is often framed in terms of flipping a weighted coin and asking how many flips one should expect before getting heads for the first time. It applies to any situation in which the probability of success is constant. In particular, it will help in our situation, where on each turn there are two possible outcomes: Either a player wins on that turn by getting the precise spin that she needs to land on square n or she does not. If the probability of success is the same on every turn of the game, then we can use the theorem above.

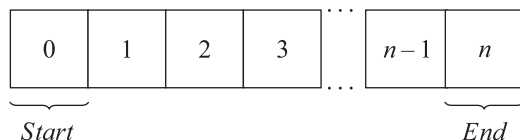


Figure 1. An n -square board without chutes or ladders.

If the board has n squares and we are playing with a spinner that has an equal probability of getting any integer value between 1 and n , then beginning with the very first turn we can reach the last square. Note that we start at a “square zero” that is before square one. Moreover, no matter where we are between square 0 and square $n - 1$, we will always win the game with probability $\frac{1}{n}$. It therefore follows from Theorem 2 that the expected number of spins to finish the game is n .

If we instead play the game with a spinner with range $n - 1$, then it is impossible to win on the first turn. However, after the first turn, the game piece will be on the board and therefore is at most $n - 1$ squares from the end. At that point, the player wins with probability $\frac{1}{n-1}$. It follows from Theorem 2 that the expected number of turns *after the first* is $n - 1$. Thus, the expected number of turns from the beginning of the game is $1 + (n - 1) = n$.

In particular, we note that we have proven Theorem 1 in the case where there are no chutes or ladders on the board.

One might wonder if in fact the expected game length does not depend on the spinner range at all; indeed, if the spinner has range 1 so that in each turn you always move exactly one space, then it is clear that the game length is exactly n . We can see, however, that this is not always the case by considering games with a spinner with range $n - 2$ where $n > 3$. In this case, it is impossible for a player to win on the first turn, but if she spins anything but a 1 on the first spin, then she will then always have probability $\frac{1}{n-2}$ of winning. On the other hand, if she does spin a 1 on her first turn, then she cannot win on the second spin but after that will always have probability $\frac{1}{n-2}$ of winning thereafter. The expected number of spins is then

$$1 + \frac{n-3}{n-2}(n-2) + \frac{1}{n-2}(1 + (n-2)) = n - 1 + \frac{1}{n-2} < n,$$

so a game played with a spinner range $n - 2$ will on average be shorter than one played with range 1, $n - 1$, or n . Even in the case of no chutes or ladders, it is an interesting and open question to see what spinner range gives the shortest game.

A single ladder to the last square

Notice that the proof of the special case of Theorem 1 in the previous section carries through identically where we have chutes and ladders on our board, as long as none of the ladders lead to the target square n . In particular, even in this more general case we have probability $\frac{1}{n}$ to win with a range n spinner and, after the first spin, we have probability $\frac{1}{n-1}$ to win with a range $n - 1$ spinner. The question is more interesting when one of the ladders takes you immediately to the final square (as is the case in the actual game) since the probability of winning on a given turn then changes depending on whether your piece is above or below the bottom rung of that ladder.

In this section, we consider the somewhat more complicated situation that arises when there is a single ladder on the board reaching from some square k to the final square. We assume that this is the only chute or ladder on the board, although our

arguments will carry through immediately as long as there is no chute “crossing” the ladder by starting at a square whose number is greater than k and ending at a square whose number is less than k . This situation is more complicated because the probability of winning on a given spin is no longer constant. In particular, if your piece is on a square below k then there are two spins that will result in a win, while if you are at a square between k and n then there is only a single winning spin.

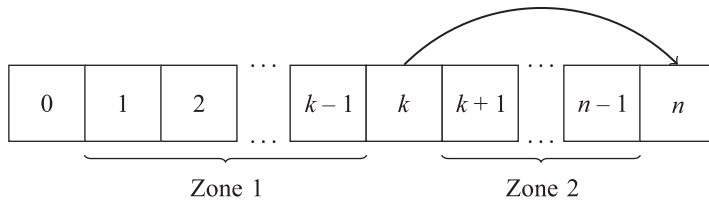


Figure 2. An n -square board with a single ladder from k to n .

We will prove Theorem 1 in this case by dividing the board into two zones, one consisting of all squares numbered between 1 and $k - 1$ and the other consisting of squares whose number is between $k + 1$ and n .

Theorem 3. For an n -square board with a ladder from some square k to square n and no chutes or ladders crossing from before k to after k , or vice versa, the expected number of turns needed to get a game piece from square 0 to square n is $\frac{n(n-k)}{n-k+1}$ for both a spinner of range n and a spinner of range $n - 1$.

Proof. For convenience, we set $x = k - 1$ to be the number of squares in zone 1 and $y = n - k - 1$ to be the number of squares in zone 2. Also, let E_1 be the expected number of turns until a player wins if she begins at any square in zone 1 and uses a spinner of range n . Similarly, E_2 for beginning at any square in zone 2. We define F_1 and F_2 similarly for a spinner of range $n - 1$.

In particular, if a game piece begins in zone 2, then we are in the same situation as if there are no ladders since there is always a unique spin that will allow the player to win. Therefore, Theorem 2 allows us to immediately conclude that $E_2 = n$ and $F_2 = n - 1$.

To calculate E_1 , we consider that a spin will take us to either zone 1, zone 2, or to square n . Because we know the expected number of spins after that first spin, we obtain a formula for E_1 in terms of E_1 and the known E_2 . Explicitly, a game piece in zone 1 has two squares that it can land on in order to win (squares k and n), so the probability of winning in one turn is $\frac{2}{n}$. Each of the y squares in zone 2 is attainable from any position in zone 1 by a unique spin, so the probability of landing in zone 2 after one spin is $\frac{y}{n}$. The remaining x spins will result in a game piece remaining in zone 1, either by moving to a square before k or by overshooting square n and therefore remaining on its current square. Combining the three cases,

$$E_1 = \frac{2}{n}(1) + \frac{y}{n}(1 + E_2) + \frac{x}{n}(1 + E_1)$$

$$E_1 = \frac{2}{n} + \frac{y}{n}(1 + n) + \frac{x}{n}(1 + E_1)$$

$$\left(1 - \frac{x}{n}\right) E_1 = \frac{x + y + 2}{n} + y$$

$$E_1 = n \left(\frac{y + 1}{y + 2} \right).$$

A similar approach shows that $F_1 = (n - 1)\frac{(y+1)}{(y+2)}$. While these results give us the expected length of a game when starting at any square on the board, the question we are interested in is the expected length of a game starting at the square 0 before the board. When using a spinner of range n , this is exactly the same question as starting in zone 1, as every square is immediately attainable. In particular, $E = E_1$.

However, when using a spinner of range $n - 1$, the final target is not attainable on the initial spin, so we need to consider things somewhat differently. In particular, from square 0 we see that there is a probability $\frac{1}{n-1}$ of landing on square k and climbing the ladder to an immediate victory, probability $\frac{x}{n-1}$ of landing in zone 1, and probability $\frac{y}{n-1}$ of landing in zone 2. Therefore,

$$\begin{aligned} F &= \frac{1}{n-1}(1) + \frac{x}{n-1}(1 + F_1) + \frac{y}{n-1}(1 + F_2) \\ &= \frac{1}{n-1} + \frac{x}{n-1} \left(1 + \frac{(n-1)(y+1)}{(y+2)} \right) + \frac{y}{n-1}(1 + n - 1) \\ &= \frac{n(y+1)}{(y+2)} = E. \end{aligned}$$

This proves Theorem 3, and in particular gives another case of Theorem 1. ■

Any number of chutes and ladders

We are now ready to consider the general case, where we have any number of chutes and ladders. In particular, assume that the total number of chutes and ladders on the board is s and that they start at the squares numbered $\ell_1 < \ell_2 < \dots < \ell_s$. We divide the game board into zones by letting zone 1 be all squares numbered strictly less than ℓ_1 , zone i the squares whose numbers are strictly between ℓ_{i-1} and ℓ_i for $1 < i \leq s$, and zone $s + 1$ all squares numbered greater than ℓ_s .

To set some additional notation, let $x_i = \ell_i - \ell_{i-1} - 1$, the number of squares in zone i . Also, let $\sigma(i)$ be the zone that the chute or ladder beginning at square ℓ_i ends at. Define the $(s + 1) \times (s + 1)$ matrix Q with q_{ij} equal to the probability moves from zone i to zone j using a range n spinner.

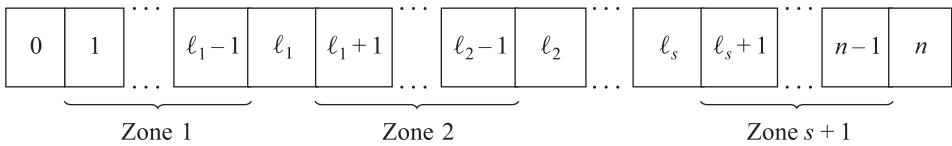


Figure 3. Splitting a board into zones.

As in the previous section, let E_i be the expected number of turns to win if the player begins at a square in zone i and uses a spinner with range n .

Proof of Theorem 1. The key observation is that, for all i ,

$$E_i = 1 + \sum_{j=1}^{s+1} q_{ij} E_j.$$

To compute the q_{ij} , we work through the possibilities of where a game piece in zone i can go. For each $j > i$, the piece will move immediately to a square in zone j (without any chutes or ladders) with probability $\frac{x_j}{n}$. Moreover, for each $j \geq i$, it will land on square ℓ_j with probability $\frac{1}{n}$ and therefore move to zone $\sigma(j)$. There is a unique spin that will allow the game piece to immediately land on the target square. In all other cases, it will stay in zone i . Putting this together, the E_i satisfy

$$E_i = 1 + \sum_{j>i} \frac{x_j}{n} E_j + \sum_{j\geq i} \frac{1}{n} E_{\sigma(j)} + \frac{i-1 + \sum_{j\leq i} x_j}{n} E_i$$

which can be rewritten as the linear equation

$$\left(n - i + 1 - \sum_{j\leq i} x_j \right) E_i - \sum_{j>i} x_j E_j - \sum_{j\geq i} E_{\sigma(j)} = n.$$

We have a system of $s + 1$ linear equations that the $s + 1$ variables E_1, \dots, E_{s+1} must satisfy. Moreover, the matrix of coefficients in this system is $n(I - Q)$ where I is the identity matrix and Q is the matrix defined above.

Since Q gives the probabilities of moving between the zones in a single spin, Q^n gives the probabilities of moving between the zones after n spins. The nature of the Markov process tells us that, as n grows large, the probability of reaching the target square (which is not a part of a zone) approaches one, so the matrix Q^n approaches the zero matrix. This in turn implies that there are no nonzero vectors v such that $Qv = v$, as any such (eigen-)vector would also have the property that $Q^n v = v$ for all n . In particular, we see that $(I - Q)v = 0$ has no nontrivial solutions, which is enough to prove that $(I - Q)$ is an invertible matrix (see [5, §11.2] for more details). For our purposes, it suffices to note that the matrix is invertible and therefore there exist unique choices of the E_i satisfying the system of linear equations.

As in the previous section, since we are assuming a spinner of range n , we may treat the starting square 0 as part of zone 1 so that $E = E_1$.

If we instead use a spinner of range $n - 1$ and define F_i to be the expected length of a game starting in zone i with such a spinner, then we can see that we get a similar set of linear equations. In particular, the exact same argument gives

$$\left(n - i + 1 - \sum_{j\leq i} x_j \right) F_i - \sum_{j>i} x_j F_j - \sum_{j\geq i} F_{\sigma(j)} = n - 1.$$

Note that each of these equations is the constant $\frac{n-1}{n}$ multiplied by the corresponding equation in the E_i . In particular, this system will also have a unique solution, specifically

$$F_i = \frac{n-1}{n} E_i.$$

That is, from any given square on the board we will expect it to take slightly longer to win the game if one uses a spinner of range n compared to a spinner of range $n - 1$. However, the game does not begin at a square on the board; while it is true that with a spinner of range n we may treat square 0 as a part of zone 1, this is not the case for

a spinner of range $n - 1$. In particular, we must make the following adjustment to our calculations:

$$\begin{aligned}
 F &= 1 + \sum_{j \geq 1} \frac{x_j}{n-1} F_j + \sum_{j \geq 1} \frac{1}{n-1} F_{\sigma(j)} \\
 &= \left(1 + \sum_{j > 1} \frac{x_j}{n-1} F_j + \sum_{j \geq 1} \frac{1}{n-1} F_{\sigma(j)} + \frac{x_1 - 1}{n-1} F_1 \right) + \frac{1}{n-1} F_1 \\
 &= F_1 + \frac{1}{n-1} F_1 = \frac{n}{n-1} F_1 = E_1 = E. \quad \blacksquare
 \end{aligned}$$

For any given *Chutes and Ladders* board, it would be a straightforward linear algebra exercise to write down the above system of linear equations and compute the values of the E_i (and thus F_i). In many cases, the system could be simplified as some of the zones will be redundant; for example, a chute or ladder that does not cross any other does not affect the calculations. However, for our purposes we needed just the existence of a solution rather than finding it explicitly. From our computations, there does not appear to be a simpler general formula for the value of E given the arbitrary nature of the board, although we suspect there may be interesting patterns for future researchers to pursue.

We note that similar techniques can be used to see how other spinner ranges affect the overall length of the game but doing so requires splitting the board up into an increasingly large number of zones, which in turn diminishes the relationships between the systems of linear equations. We leave these questions for future readers to explore and look forward to seeing their results in future *College Mathematics Journal* issues!

Summary. We prove a conjecture from a 2011 *College Mathematics Journal* article addressing the expected number of turns in a *Chutes and Ladders* game when the spinner range is close to the length of the board. While the original paper approached the question using linear algebra and the theory of Markov processes, our main method uses combinatorics and recursion.

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