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Abstract
Searching for a job is always stressful and, with unemployment rates at their highest levels in years, never more so than now. Applicants can and should use every advantage at their disposal to obtain a job which is rewarding, financially and otherwise. While this author believes a math major gives applicants many advantages as they search for their dream job, one often overlooked is the ability to strategize and schedule their interviews to maximize the chance of landing that job.

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Disciplines
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The Secretary Problem from the Applicant's Point of View

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Searching for a job is always stressful and, with unemployment rates at their highest levels in years, never more so than now. Applicants can and should use every advantage at their disposal to obtain a job which is rewarding, financially and otherwise. While this author believes a math major gives applicants many advantages as they search for their dream job, one often overlooked is the ability to strategize and schedule their interviews to maximize the chance of landing that job.

The secretary problem helps an employer find the optimal candidate for a job out of a large pool of applicants. The set up is as follows: only one person can be hired, and, for any pair of applicants, the employer has a strict preference for one of them that they can discern after seeing both. However, after each interview the employer must either accept or reject the candidate. If the candidate is the final person, the interviewer simply must accept them, as rejected candidates cannot be recalled. In the classical formulation of the problem, the goal is to select the best applicant overall.

What strategy can the employer use to maximize the probability of hiring the best overall applicant? It is clear that, other than the final candidate, you only hire an applicant if they are the best applicant you have seen to that point. Otherwise you are certainly not hiring the best person. It follows that the best strategy is to reject an initial number of candidates and then hire the first candidate who is better than everyone seen so far. With a little work, one can show that if you know there will be a total of n candidates, where n is large, then you should initially reject $k \approx \frac{n}{e}$ candidates. For a proof, see [**3**]. A more recent article discussing this problem and its implications for students' dating lives can be found in [**6**].

While there is some uncertainty regarding the origin of this problem, its first published appearance was in Martin Gardner's Mathematical Games column in February of 1960 (reprinted in [**2**]). A detailed history of the secretary problem and other classical stopping problems is in [**1**].

The secretary problem is an outstanding example of how a seed planted by one of Gardner's columns has grown and flourished. Over the intervening half-century, people have studied many variants. For example, how does the answer change if the

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interviewer is allowed to recall the last few applicants [**7**]? Or what if there is a full committee of interviewers rather than a single interviewer [**4**]? Or how should one deal with the costs of interviewing and pressures to hire an early applicant to reduce costs [**5**]? As of this writing, there are over 130 papers in MathSciNet referring to the Secretary Problem and its extensions.

Almost all published variants look at the problem from the employer's perspective. At a recent senior thesis presentation on one such variant at Gettysburg College, another senior, clearly concerned about his own job prospects, asked "What does this mean for applicants? If I know that an employer is going to behave optimally, when should I schedule my interview?" This is the question we examine here.

When rank is unknown

First we consider the situation of an applicant who has no idea how strong they are compared to the rest of the pool. Throughout this note, we assume that there are n applicants for a single job, and that the employer's strategy is initially to reject the first k and then hire the first subsequent candidate who is better than all candidates already seen, if such a candidate exists, or the final candidate otherwise. We call this the *optimal strategy*. We assume that all $n!$ possible ordered rankings of the n candidates are equally likely and examine the probability that each position in the interview order is the one where the chosen applicant is found.

Let us define the random variable X to be the interview position of the applicant who is finally chosen. Using the optimal strategy, the employer rejects the first k applicants, and therefore $P(X = i) = 0$ if $1 \leq i \leq k$. In order for the $(k + 1)$st candidate to be chosen, they must be better than all the candidates in the rejected group. This occurs exactly when the $(k + 1)$st candidate is the best among the first $k + 1$ candidates, which occurs with probability $\frac{1}{k+1}$. More generally, for $k + 1 \leq i \leq n - 1$, the ith candidate will be chosen if and only if:

- The ith candidate is the best of the first i candidates.
- The best of the first $i - 1$ candidates is in the initial rejected group of k candidates.

The second condition ensures that we get to the ith candidate without choosing someone else; the first condition ensures that we then choose the ith candidate. The probability of these two conditions simultaneously holding is $\frac{1}{i} \cdot \frac{k}{i-1}$.

Finally, there are two separate situations in which the *last* candidate is selected: either the best candidate overall was in the initial group of rejected candidates, in which case the final candidate is chosen as a last resort, or the best overall candidate is the last one interviewed and the *second* best candidate was in the initial group of rejectees. The first scenario occurs with probability $\frac{k}{n}$; the second with probability $\frac{1}{n} \cdot \frac{k}{n-1}$. Adding these two cases together, $P(X = n) = \frac{k}{n} + \frac{k}{n(n-1)} = \frac{k}{n-1}$. In summary, we have the following result.

Theorem 1. *The probability that the* ith *candidate is chosen is*

$$P(i) = \begin{cases} 0 & \text{if } i \leq k, \\ \frac{k}{i(i-1)} & \text{if } k < i < n, \\ \frac{k}{n-1} & \text{if } i = n. \end{cases}$$

The goal of the applicant is to obtain interview slot i which maximizes $P(i)$. Note that $\frac{k}{i(i-1)}$ is a decreasing function in i. Therefore the candidate should target the $(k+1)$st slot or the final slot, depending on whether $\frac{1}{k+1}$ or $\frac{k}{n-1}$ is larger. An applicant will prefer the final slot if $\frac{k}{n-1} > \frac{1}{k+1}$. This is the case when

$$k > \frac{-1+\sqrt{4n-3}}{2}, \tag{1}$$

agreeing with the intuition that rejecting a larger number of candidates makes it more likely that the best candidate is in the rejected group and that the employer accepts the final candidate.

If the employer uses the optimal strategy with $k = \frac{n}{e}$, then the applicant must check whether (1) holds, which happens when $k^2 + (1-e)k + 1 > 0$. A simple calculation shows this is always the case, and therefore the candidate should choose to be interviewed in the final slot if they have no information about their relative standing with the other candidates. Of course, $\frac{n}{e}$ is never an integer, so in reality the employer chooses either $k = \lfloor \frac{n}{e} \rfloor$ or $\lceil \frac{n}{e} \rceil$. In the latter case, the candidate will still wish to choose the final interview slot, because if $\frac{n}{e} > \frac{-1+\sqrt{4n-3}}{2}$ then certainly $\lceil \frac{n}{e} \rceil$ is as well. If the employer rounds down then one can check that the candidate will be better off choosing last as long as $k > 2.5$. In particular, being interviewed in the final position is optimal if $n \geq 10$. One can manually check that, if the interviewer is going to reject $k = \lfloor \frac{n}{e} \rfloor$ applicants, one will be no worse off being interviewed last except when $n = 4$, 5, or 8. Thus we have proved

Theorem 2. *If the number of applicants to a position is at least nine and an employer uses the optimal strategy then the probability that they hire the* final *person interviewed is higher than any other person.*

When rank is known

On the whole, in the absence of information about their standing, a candidate should choose to be interviewed last. This is also true if you know you are the worst candidate, as you will never be better than all the candidates in the rejected group, so you can *only* be chosen if you are the last resort. On the other hand, if you know you are the strongest candidate, then you prefer the $(k+1)$st slot over all others. You are then guaranteed to be chosen: you are certainly better than anyone in the rejected group, and in any later slot there is a chance that someone else is picked before you are interviewed. In general, then, it seems that stronger applicants want to be interviewed early and weaker applicants later. In this section, we prove that this is so by considering the point of view of a candidate who knows they are the jth best in a pool of n candidates.

Assume first that our candidate interviews last. As in the previous section, if $j \neq 1$ then the only way that they can be chosen is if the best candidate is in the rejected group, which happens with probability $\frac{k}{n-1}$, since there are $n-1$ slots remaining for the best applicant to be in, all equally likely. On the other hand, if $j = 1$ then the only way to be chosen from the last slot is if the second best candidate is rejected, which also happens with probability $\frac{k}{n-1}$. In other words, the probability that the candidate in the final slot is chosen is $\frac{k}{n-1}$ independent of how strong they are.

As before, if a candidate is interviewed in slot i, $k < i < n$, they will be chosen if and only if they are the best candidate seen so far and the second best candidate to that point is rejected. Assume that $i > 1$, which will is the case as long as $k > 0$, i.e., at least one candidate is rejected. The probability that the jth best candidate overall is the best candidate seen so far, when they are interviewed in the ith slot, is the probability that all $i - 1$ candidates seen previously come from the $n - j$ candidates who are worse than them. This happens with probability $\frac{\binom{n-j}{i-1}}{\binom{n-1}{i-1}}$. Moreover, the probability that the best candidate seen before the ith slot was in the rejected group is $\frac{k}{i-1}$. Putting this together, we get the following result:

Theorem 3. *Assume the jth best candidate is interviewed in the ith slot. The probability of getting chosen is:*

$$P_j(i) = \begin{cases} 0 & \text{if } i \le k, \\ \frac{\binom{n-j}{i-1}}{\binom{n-1}{i-1}} \frac{k}{i-1} & \text{if } k < i < n, \\ \frac{k}{n-1} & \text{if } i = n. \end{cases}$$

Again, $P_j(i)$ is a decreasing function in the range $k < i < n$. Explicitly, we note that if i and $i + 1$ are in this range, then

$$\begin{aligned} \frac{P_j(i)}{P_j(i+1)} &= \frac{\binom{n-j}{i-1}}{\binom{n-1}{i-1}} \frac{k}{i-1} \frac{\binom{n-1}{i}}{\binom{n-j}{i}} \frac{i}{k} \\ &= \frac{(n-i)!\,(n-i-j)!}{(n-i-1)!\,(n-i-j+1)!} \frac{i}{i-1} \\ &= \frac{n-i}{n-i-j+1} \frac{i}{i-1} > 1, \end{aligned}$$

and therefore $P_j(i) > P_j(i + 1)$, as desired. This means that for any fixed value of j, an applicant should either choose to be interviewed in the $k + 1$st slot or in the nth slot. In order to see which of these options gives the higher probability, we must compare $P_j(k + 1) = \frac{\binom{n-j}{k}}{\binom{n-1}{k}}$ with $P_j(n) = \frac{k}{n-1}$. After some algebraic manipulation, we find that $P_j(k + 1) > P_j(n)$ if and only if $\binom{n-j}{k} > \binom{n-2}{k-1}$. We wish to consider this inequality separately for different values of j.

If $j = 1$, then we compare $\binom{n-1}{k}$ to $\binom{n-2}{k-1}$. Since $\binom{n-1}{k} = \binom{n-2}{k} + \binom{n-2}{k-1}$, the best candidate will prefer to be interviewed in the $(k + 1)$st slot, as we argued earlier.

If $j = 2$, then we compare $\binom{n-2}{k}$ and $\binom{n-2}{k-1}$. The former is larger exactly when $k < \frac{n-1}{2}$. If the employer chooses $k \approx n/e$, then this holds for large values of k, implying that the second best candidate should also choose the $(k + 1)$st slot.

If $j = 3$, then we compare $\binom{n-3}{k}$ and $\binom{n-2}{k-1}$. Looking at the quotient of these terms and expanding algebraically, we see that the former is larger exactly if

$$k^2 + (5 - 3n)k + (n^2 - 3n + 2) > 0,$$

which occurs if $k < \frac{3n-5-\sqrt{5n^2-18n+17}}{2}$. For large n, this is true if $k < \frac{3-\sqrt{5}}{2}n \approx .38n$, which is (barely) guaranteed if the employer chooses $k = n/e \approx .367n$. More specifically, one can work out that the third-best candidate should choose the $(k + 1)$st slot for all cases except for 33 specific values of n, the largest of which is $n = 98$.

For $j \geq 4$, comparing $\binom{n-j}{k}$ and $\binom{n-2}{k-1}$ is difficult. In particular, one can show that the former term is greater exactly when

$$\frac{n-k-1}{k} \prod_{\ell=1}^{j-2} \frac{n-k-\ell-1}{n-\ell-1} > 1.$$

Assuming that $n = ek$ and letting n be large, we see that this holds if and only if $\frac{(e-1)^{j-1}}{e^{j-2}} \geq 1$, which is false for $j \geq 4$. In particular, for sufficiently large n one prefers to be interviewed last if ones ranking is fourth or worse. We summarize the preceding results:

Theorem 4. *If you are the* jth *best applicant then you want to be interviewed in the* $(k+1)$st *slot for* $j \leq 3$*, and in the final slot if* $j \geq 4$ *and* n *is sufficiently large.*

We also note that for any n and $j \geq 3$ and $k > \frac{n-j}{2}$ we will have that $\binom{n-2}{k-1} > \binom{n-j}{k-1} \geq \binom{n-j}{k}$. If the employer sets $k = \lceil \frac{n}{e} \rceil$ then k will be greater than $\frac{n-j}{2}$ if $\frac{j}{n} > \frac{e-2}{e} \approx .26$. In particular, this says that regardless of n, if a candidate suspects they are not in roughly the top quarter of candidates then they would prefer to be interviewed in the last slot.

Final thoughts

Unless you are a truly extraordinary applicant, if your prospective employer uses the classic strategy, then you should try to be interviewed last. However, attempting to game the system is likely to backfire as there is no margin of error. Being the final interviewee gives one the highest probability of being hired, but being the second-to-last has one of the lowest! Students going on the job market should put their energy into improving their resumés rather than strategizing interview timing. We note that a student can do *both* by generalizing our work to some of the variants of the secretary problem mentioned earlier—and publishing their work in future issues of this JOURNAL.

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Summary. A 1960 "Mathematical Games" column describes the problem, now known as the Secretary Problem, which asks how someone interviewing candidates for a position should maximize the chance of hiring the best applicant. This note looks at how an applicant should respond, if they know the interviewer uses this optimal strategy. We show that all but the very top applicants have the best chance of being hired if they arrange to be the last person interviewed.

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Crossword Solution

(puzzle on pp. 70–71)

■	M	A	R	T	I	N	■	G	A	R	D	N	E	R	■	■
H	I	T	T	U	N	E	■	E	M	I	R	A	T	E	S	■
R	E	V	E	R	S	I	■	R	A	C	E	T	R	A	C	K
E	N	S	■	B	E	G	S	■	N	E	V	■	E	L	A	N
■	■	■	G	O	T	H	I	C	■	D	I	N	■	I	L	O
B	A	S	E	■	■	■	C	O	L	■	L	O	S	T	I	T
A	S	P	E	C	T	S	■	N	A	G	■	S	O	Y	A	S
L	E	O	■	H	A	H	■	C	I	R	C	U	S	■	■	■
M	A	T	H	E	M	A	T	I	C	A	L	G	A	M	E	S
■	■	■	O	E	U	V	R	E	■	B	O	A	■	A	N	A
G	H	O	S	T	■	E	E	R	■	S	P	R	O	U	T	S
R	E	R	E	A	D	■	U	G	H	■	■	■	K	I	S	S
O	R	D	■	H	E	X	■	E	I	D	E	R	S	■	■	■
O	M	I	T	■	J	A	N	■	P	E	N	A	■	N	I	M
M	I	N	O	R	A	X	I	S	■	F	E	M	B	O	T	S
■	T	E	L	E	V	I	S	E	■	O	R	B	I	T	A	L
■	■	E	L	E	U	S	I	S	■	G	O	O	G	O	L	■