Winter 2009

Invariance: A Tale of Intellectual Migration

Steven Gimbel
Gettysburg College

Gimbel, S. (2009). Invariance: A Tale of Intellectual Migration. Convergence Review 1(1): 41-60.

Invariance: A Tale of Intellectual Migration

Abstract
The plotline of the standard story told about the development of intellectual history at the end of the 19th/turn of the 20th century follows the move from absolutism to perspectivalism. The narrative takes us, on the one hand, from the scientism of late Enlightenment writers like Voltaire, Mill, D'Alebert, and Comte and the historical determinism of Hegel, all of which were based upon a universal picture of rationality, to, on the other hand, the relativistic physics of Einstein, the perspectival art of Picasso, and the individualism of Nietzsche and Kierkegaard leading to the phenomenology of Husserl and Heidegger to and on through the deconstructivist work of Derrida in which universal proclamations were deemed meaningless. In their place, was relative dependent upon subjective, political, and social factors, influences, and interpretations. Like all sketches, of course, the story is more complicated than that.

There is another trend in the intellectual air of the early 20th century that gets left out of this oversimplified picture, one that threads a middle path between absolutism and perspectivalism, a path that considers both frame-dependent or covariant truths and frame-independent or invariant truths and examines the relations between them. Indeed, the notions of covariance and invariance play important roles in the development of the fields of mathematics, physics, philosophy, and psychology in the decades after the turn of the 20th century.

The migration of the concepts of invariance and covariance illustrates not only the interconnectedness of the working communities of intellectuals, but also displays ways in which the personal, social, and political overlaps between groups of disciplinary thinkers are essential conduits for the conceptual cross-fertilization that aids in the health of our modern fields of study. [*excerpt*]

Keywords
invariance, covariance, absolutism, perspectivalism, scientific truth

Disciplines
Intellectual History | Philosophy | Philosophy of Science

STEVE GIMBEL

Invariance: A Tale of Intellectual Migration

The plotline of the standard story told about the development of intellectual history at the end of the 19th/turn of the 20th century follows the move from absolutism to perspectivalism. The narrative takes us, on the one hand, from the scientism of late Enlightenment writers like Voltaire, Mill, D'Alebert, and Comte and the historical determinism of Hegel, all of which were based upon a universal picture of rationality, to, on the other hand, the relativistic physics of Einstein, the perspectival art of Picasso, and the individualism of Nietzsche and Kierkegaard leading to the phenomenology of Husserl and Heidegger to and on through the deconstructivist work of Derrida in which universal proclamations were deemed meaningless. In their place, was relative dependent upon subjective, political, and social factors, influences, and interpretations. Like all sketches, of course, the story is more complicated than that.

There is another trend in the intellectual air of the early 20th century that gets left out of this oversimplified picture, one that threads a middle path between absolutism and perspectivalism, a path that considers both frame-dependent or covariant truths and frame-independent or invariant truths and examines the relations between them. Indeed, the notions of covariance and invariance play important roles in the development of the fields of mathematics, physics, philosophy, and psychology in the decades after the turn of the 20th century.

The migration of the concepts of invariance and covariance illustrates not only the interconnectedness of the working communities of intellectuals, but also displays ways in which the personal, social, and political overlaps between groups of disciplinary thinkers are essential conduits for the conceptual cross-fertilization that aids in the health of our modern fields of study.

Late 19th Century Algebra and the Introduction of Covariance and Invariance

The theory of invariants is invariably associated with the famed British mathematical pair Arthur Cayley and J.J. Sylvester, who developed the theory while working as lawyers. Both displayed great talent for mathematics while young, yet both ended up in law because the British professorate at the time required Anglican priesthood. This line was not available to Sylvester, who was Jewish, and was refused by Cayley who, while an orthodox Anglican of undeniable faith, did not feel a call to the priesthood and so could not in god conscience take Holy Orders for mere personal advantage.[1]

This sense of earnestness was a trait Cayley displayed through all of his life. Sylvester, on the other hand, was a man in search of what he deemed to be legitimate recognition for a mind of his outstanding caliber and this led him to fight vigorously whenever he felt he had been slighted in personal or intellectual matters,[2] even trivially so, and to work on open mathematical problems of massive historical significance.[3]

Sylvester's fire to Cayley's ice allowed the pair to work fruitfully on problems across the mathematical spectrum, but the place where their efforts had the most impact was in the field of algebra.

Algebraists in the 19th century were studying the way the forms of equations changed under transformation of variables. Equations posit relations between quantities. These quantities are written in terms of coordinates that are freely chosen linguistic conventions. Some choice of coordinates must be made in order to have a language in which to express the relations. The variables themselves are meaningless empty names, but once a coordinate frame is selected, one may meaningfully make statements about relations between them. You can translate these relations between choices of coordinates, just as one may translate the sentence "Snow is cold" into German or French, but some language is needed and this requires a choice of coordinates. Certain choices may make particular calculations easier than others, and so it is often advantageous to transform the equations by replacing one variable selection with another, but ultimately it is a completely free choice what coordinates are invoked.

The French mathematicians Joseph-Louis Lagrange and Pierre-Simon Laplace, the Germans Friedrich Gauss and Gotthold Eisenstein, and the Englishman George Boole had all investigated interesting cases in which, for certain classes of equations, certain properties of the structure of the equations and their solutions remained the same before and after transforming their variables by choosing different coordinate systems. Cayley termed these unchanging properties, "hyperdeterminants," but Sylvester – who wrote of himself, "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all other mathematicians of the age combined"[4] – coined a new name for them, "invariants."

Upon reading Boole's paper of 1841, "Exposition of a General Theory of Linear Transformations," Cayley became fascinated with the notion of invariants and began a correspondence with Boole in which he became determined to ferret out all the invariants of the binary quadratic and binary quintic forms and, more importantly, to find a systematic way of accounting for them. Some invariants, it turned out were special cases of more general invariants, and Cayley was determined to find the complete basis set of these invariants, that is the smallest set of invariants that could be used to construct all possible invariants associated with those forms of equations. Cayley and Sylvester worked on this problem throughout their lives, but they made their historic strides in the 1840s and 1850s, giving rise to invariant theory in the form we know it today.

Once they had begun their great work in earnest, Cayley came to realize that their new theory of invariants was not just a curiosity of certain algebraic equations. Rather, it had a deeper meaning.

> This idea of 'permanence among change' has been such a recurring idea in religious, philosophic, and scientific thought that it's explicit appearance in mathematics would have acted as a magnet to the

> young mathematician on the lookout for new subjects. Had he not assimilated the Platonic vision of scientific research as taught at King's College, London? Classics, supervised by R. W. Browne at King's, taught that Plato's interpretation of the object of science as the search for 'the true, the eternal, the immutable" was the correct vision and that to succeed in this cause 'is to know intellectually the essence of things absolutely.'[5]

Not only did invariants expose something meaningful within algebra, but it was a Platonic tool to be used to revolutionize fields throughout mathematics.

The Migration of Invariance into 19th/20th Century Geometry

When meeting Felix Klein, one

> could not have avoided being impressed. Klein was a tall, handsome, dark haired and dark bearded man with shining eyes, whose mathematical lectures were universally admired and circulated even as far as America.[6]

Widely considered second only to the great Henri Poincaré as the most important mathematical mind in the first half of the 20th century, it could be said of Klein, like Cayley, that "his mathematical interest was all-inclusive. Geometry, number theory, group theory, invariant theory, algebra."[7] Also, like Cayley, he was one to draw connections between the seemingly distinct mathematical subfields.

Traditionally, mathematics had two seemingly distinct areas of study: algebra and geometry. By the time Cayley was helping to revolutionize algebra, geometry had already commenced a radical transformation. Ivan Lobachevski's created hyperbolic geometry, a new geometric system whose basic axioms were inconsistent with those of Euclid, whose work *The Elements* had shaped the field since classical times and were thought by the greatest minds of history to be unassailable. Lobachevski's work struck at the core of confidence in mathematical truth. Mathematics, it was held, was the one place where definite knowledge was available to humans. From Plato to Descartes, mathematics, particularly the rigorous proofs of Euclidean geometry, provided the model for the finished state of all other human intellectual endeavors. But the specter of a competitor to Euclid meant that perhaps the certainty that was attributed to geometry could not be counted upon. If we could not fall back on Euclid for certainty, then perhaps nothing at all was certain. But this seemed absurd.

As absurd as Lobachevski's new geometric system. In the space Lobachevski describes, it is impossible to have figures of different sizes with the same internal angles. There could be no rectangles. The internal angles of a triangle were always less than two right angles. It all seemed so bizarre, so counter to our most basic intuitions and so surely had to be false. In mathematics, that means that it would have to give rise to a contradiction. But while more and more fantastic theorems could be derived, no inconsistency arose. The search continued on for if even a single contradiction could be shown to follow from the axiom set of this new geometry, then it could be rejected wholesale, the supremacy of Euclid reestablished, and the basis for belief in classical thought reasserted with complete confidence. All hopes for rescuing the old order rested on this hope.

But Klein, along with Poincaré and Eugeno Beltrami, showed that the superiority of Euclid could never be established. Klein gave the basic geometric terms new meanings and showed that Lobachevski's axioms, when understood in this new, non-standard fashion, were true of a set of Euclidean objects. Since a sentence is a contradiction by virtue of its form (the sentence "I have a blim and I don't have a blim" is false no matter what I decide to mean by the word "blim"), by cleverly translating the non-Euclidean axioms into the Euclidean language so that the reinterpreted non-Euclidean axioms became deductively necessary results of Euclid's own system, Klein demonstrated that the only way Lobachevski's axioms could imply a contraction is if the Euclidean axioms also implied a contraction. No one wanted to say that Euclid was flawed, so on the basis of this relative consistency proof, no one could not say that the non-Euclidean geometry was flawed either.

But Lobachevski's system was not the only new geometry. Indeed, geometric systems proliferated at a stunning rate. More conservative geometers refused to even consider such studies geometry. Klein, to his detriment, did not belong to such a group.

> [Klein] was much younger than his colleagues, and they resented his innovating tendencies. In particular, there was opposition to his determination to avail himself of the vaunted German 'Lehrfreiheit,' and to interpret the word "Geometry" in its widest sense.[8]

Klein embraced the non-standard geometries, but this proliferation came with a cost, the field lost its elegance and that is a property mathematicians revere. Something had to be done to tame the wildness emerging in geometry, to bring order and structure to the multiplicity of systems. The concept that could be used to do the job was invariance and it was Cayley himself who first realized it.

Among the more abstract systems to arise was projective geometry. Think of trying to make a flat map out of a globe. To do this, the landmasses on the globe would have to be projected onto a flat screen and traced out. But in this tracing process, the move from the curved surface of the globe to the flat surface of the screen would distort size, shape, and angles – consider, for example the way that countries far from the equator, like Greenland, are significantly larger on the standard Mercator projection maps than they ought to be. Size in such a projection is not invariant. But there are geometric relations that do remain invariant and Cayley realized that these relations form what mathematicians call a group, that is, it is a closed set with an operator that when applied to any member of the set will point you to another member of the set.

Klein first read of Cayley's work in George Salmon's textbooks[9] that were widely translated and were one of the major routes through which English mathematics was channeled to the Continent. Salmon had been a friend and constant correspondent of Cayley and Sylvester and his four careful, detailed textbooks were well-studied and appreciated in his time. Cayley's application of invariant theory coupled with group theory had opened a door to understanding geometry in a much broader sense, but the intellectually conservative Cayley refused to admit what was in the room behind it.

He denied the legitimacy of the new geometric systems, even though he realized that in some way invariant theory would allow us to make sense of them.

Klein, on the other hand, had no such problem and when he received his first significant academic post at the university in Erlangen, he used the occasion of his inaugural lecture to set out what became known as the *Erlangen Programm*, the project of accounting for the interrelatedness of all geometric systems in terms of associated groups and invariants. If one system had a group of invariants that was a subgroup of the invariants of another system, one could see the relation between the two.

Klein continued to expand his work in this direction through his positions at Erlangen, Munich, Leipzig, and ultimately Göttingen, a beautiful, pastoral university that he transformed into the mathematical Mecca. But this work was not done in isolation.

> In looking through Klein's Autobiography, one is struck by his consistent references to his friendships. It was characteristic of him that they all bore directly, or indirectly, on his mathematical development and on his power of organization. He was indeed a man without a hobby; in particular, and this is curious and interesting from a psychological point of view, although a German, and although endowed with an excessive acuteness of hearing, he could not distinguish one tune from another.
>
> Social intercourse for Klein meant the interchange of ideas, and for that he was as eager as the ancient Athenians. It was in such give-and-take that his own conceptions took form. There is, perhaps, no contradiction in saying that Klein was never the originator of his own ideas. He had not the generating force of a Cauchy or a Georg Cantor, but he had a phenomenal power of grasping the import of a suggestion, and working it out on a grander scale than any before him had imagined.[10]

Klein was a social thinker who depended on interaction to cross-pollinate his thoughts. One such occasion was afforded when Klein was in Munich and Cayley, with whom he had been corresponding, spent the three months of the summer of 1880 visiting the Kleins.

While Cayley never accepted the breadth of Klein's geometric work, the result – now called Cayley-Klein geometry – shaped and structured the way the field advanced. The notion of invariance became an essential mathematical tool beyond its algebraic origins.

Invariance and the Theory of Special Relativity

Klein had a bright young student named Adolf Hurwitz who followed him from Munich to Leipzig where he received his Ph.D. under Klein and ultimately secured a university post at Königsberg, an isolated country outpost in the German academy, but a pleasant place with a long and treasured intellectual history.

Here, he found himself with a pair of young, but very inspired and incredibly talented students. One, David Hilbert, would go on to be one of the most famed minds in the history of mathematics and would be the co-discoverer of the general theory of relativity, developing it at the same time – although along a different methodological path – as Albert Einstein. The other, Hermann Minkowski, would also work in both pure mathematics and mathematical physics.

While Hilbert was an out-going, social person who loved dancehall music and was an interminable flirt, his dear and life-long friend Minkowski was bookish and shy. Minkowski's family was Russian Jewish and had moved west to Königsberg because of the oppression of Jews under the Czar. It was an experience that shaped Minkowski. After having to sell all of the family's possessions, including their books, Minkowski memorized the works of Goethe so that should he again end up without books, he would never be without literature.[11] Hilbert's mind, like his personality, was broad and explosive, while Minkowski was much more rigorous and thorough. The three would meet regularly under an apple tree near the university and take long walks arguing and exploring topics and open questions covering the entire span of mathematics.

Hilbert's doctoral dissertation topic was an extension of Cayley's theory of invariants and a copy was sent to Minkowski who responded,

> I studied your work with great interest, and rejoiced over all the processes which the poor invariants had to pass through before they managed to disappear. I would not have supposed that such a good mathematical theorem could have been obtained in Königsberg![12]

It would be Minkowski, though, who would use invariant theory to make his greatest contribution.

They complimented each other and corresponded whenever they were separated. As Hilbert climbed the ladder of the German academy, moving to ever more prestigious positions, he always made sure to use his leverage to pull Minkowski into the newly vacated position. This continued until Minkowski left the German system and took a position at the Eidgenössische Technishe Hochschule in Zürich where he was to have his most famous student, Einstein.

The relationship between Minkowski and the young Einstein was not a pleasant one. In the classroom, Minkowski's shyness led him to speak quietly in a halting and stammering fashion. His pedagogical flaws led Einstein, who needed little reason, to skip many of Minkowski's lectures, relying upon the notes of his friend and classmate Marcel Grossman without whom he likely would not have made it through his university exams. Minkowski thought Einstein arrogant, impudent, and insufficiently serious, referring to him as a lazy dog. Einstein did not think any better of his professor Minkowski at the time although he was later to list him among his "excellent teachers" at the ETH.[13]

Einstein did graduate from the ETH, but his poor relationship with his professors left him without a job as an assistant despite great effort and years of trying. It would be virtually assumed that a graduate from this prestigious school, with well-connected faculty would naturally take the next step along the career path of a developing scientist, but Einstein, because of his attitude as a student was denied. Instead, Grossman again saved him when his father secured him a position as a patent clerk, a respectable civil service job. He was working this job in the miraculous year of 1905, when Einstein wrote his famous "On the Electrodynamics of Moving Bodies," the paper that introduced the special theory of relativity to the world.

The work begins in a curious fashion, considering two thought experiments. In the first, a magnet is held still and a wire coil around it, while connected to a circuit, is moved back and forth. This gives rise to an induction current. Next, the coil is held still while the magnet is moved back and forth in the same way inside of it, giving rise to an equivalent current. Under the old theory, Maxwell's electrodynamics with a field bearing, but invisible ether, the two received different physical explanations.

> Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case.[14]

From this equivalence, that is, from the invariance of the hypothetically measured electrical current in the circuit, Einstein argues that the invisible ether, a significant metaphysical postulation by advocates of classical physics, in fact does not exist.

> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the 'light medium,' suggest that the phenomena of electrodynamics as well as mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good…The introduction of a 'luminiferous ether' will prove to be superfluous inasmuch as the view here to be developed will not require an 'absolutely stationary space' provided with special properties.[15]

Einstein employs Cayley's concept of invariance under transformation here, except that instead of a mathematical entity maintaining its form under transformation of coordinates, it is a physical measurement maintaining its numerical value under a change of reference frame. The concept of invariance is key to the very notion of relativity.

While Einstein's ideas were revolutionary, not many paid attention to them. Challenging Isaac Newton's theory of mechanics which was the best confirmed scientific theory in history, standing unchallenged for three centuries was audacious enough, but to replace it with a theory so peculiar, such an affront to common sense, led many to think it a complete non-starter. The theory claimed that there was no fact in the world about the simultaneity of spatially separated events. It claimed that moving objects would be shorter in moving frames of reference than they were in frames attached to the objects. Einstein argued that the length of time passing depends upon your state of motion. It was a bizarre theory challenging the heart of classical physics as proposed by a nobody and most physicists would have none of it.

But there was one who did see the insight, the genius, the completely revolutionary nature of the theory – Hermann Minkowski, who commented about the theory, "Oh, that Einstein, always missing lectures – I really would not have thought him capable of it!"[16] Minkowski saw in Einstein's work a new picture of the world that was not completely seen at the time by anyone else, including Einstein himself because of the sort of mathematical language in which the theory was worked out. "Einstein's presentation of his deep theory is mathematically awkward – I can say that because he got his mathematical education from me at Zürich,"[17] Minkowski was known to comment having joined Hilbert and Klein at Göttingen.

In an address entitled "Space and Time" delivered to the 80th Assembly of German Natural Scientists and Physicians in 1908 at Cologne, he expressed Einstein's theory in an entirely novel way. Einstein writes,

> Minkowski's important contribution to the theory lies in the following: Before Minkowski's investigation it was necessary to carry out a Lorentz-transformation on a law in order to test its invariance under such transformations; he, on the other hand, succeeded in introducing a formalism such that the mathematical form of the law itself guarantees its invariance under Lorentz-transformations.[18]

This understates the case a bit. This formalism used by Minkowski takes the geometric work of Cayley and Klein and "geometrizes" the theory of relativity, that is, it shows how the theory forces us into a fundamentally new understanding of the universe as a four-dimensional integrated picture of space and time.

Minkowski begins the address with his famed passage,

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lays their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.[19]

It is clear that there is more than a new formalism here, more than simply a clearer linguistic framing, rather, there is a new way to understand the structure of the universe itself.

We must think of the world in which we live as a combination of three spatial dimensions and a fourth temporal dimension combined together. The points of this world are events, that is, places at a time. Objects in this space-time are like tubes flowing through it. Any slice along that tube is the object at a given time. The trail of its tube is its world-line through space-time.

Attached to each event is a light cone in the future pointing direction and one in the past pointing direction. It is called a light cone because it is formed by considering all of the points in space-time that could be reached by a signal moving at the speed of light from or to that point. Since no signal can travel faster than the speed of light, the points inside of the past pointing light cone are the set of all events that could causally effect the point, that is, the set of events in the past pointing light cone of an event contains everything about the universe that it is possible for someone at that point in space at that time to know and everything in the universe that could have any effect at all on that event. If an event is outside of the light cone, someone at that point could have no knowledge of it and it could in no way influence it. Similarly, the future pointing light cone contains all of the events in the universe that this event could have any effect upon. If an event is outside of the future pointing light cone, the original event cannot be causally related to it and an observer at that point could have no knowledge of the original event.

More than a formalism, what Minkowski provides for special relativity is a complete worldview, a metaphysic, a new image of the nature of reality itself. It was the use of the Cayley-Klein approach to geometry on Einstein's clumsily formulated mathematics that changed the way we see space and time.

General Covariance and the General Theory of Relativity

Minkowski's work completely changed the way Einstein himself understood his own theory. As he thought about the malleable space-time Minkowski sketched from his work, Einstein came to find two flaws. First, while the theory included aspects of mechanics, optics, electricity and magnetism, there was one physical force that was nowhere to be found – gravitation. The theory would have to be expanded. Second and more related to Minkowski, the invariant quantities Minkowski points out only remain invariant in a special set of reference frames, those that move in straight lines at a constant speed relative to each other. These covariant frames are called "inertial frames" and the laws of the theory held good only for them. This limitation bothered Einstein who thought that the laws of physics should be the same for all observers.

> In contrast to [the limited invariance of the special theory of relativity] we wish to understand by the 'general principle of relativity' the following statement: All bodies of reference *K, K'*, etc., are equivalent for the description of natural phenomena (formulation of the general laws of nature), whatever may be their state of motion.[20]

This limitation of the theory to the special case of inertial reference frames is why the theory is called "special relativity." Einstein wanted to make the theory universal, to make the governing physical equations of a relativistic gravitation theory hold for all possible observers, that is, to create a general theory of relativity, not just one for special reference frames.

Just as his 1905 paper began with a thought experiment in which two cases were seen to be different accounts of the same phenomenon, so too with his 1916 paper introducing the general theory of relativity. Consider the counter-intuitive fact that heavier things do not fall faster. This is a strange and unique aspect to gravitation. If you have two charged bodies and you double the charge on one, then you double the acceleration due to the resulting electrical force. It is the same with magnetic force. But if you double the mass, that is, the gravitational charge, you do not double the acceleration due to gravity.

Suppose you were in an elevator with a bathroom scale sitting still on the bottom floor of a tall office building. Stepping on the scale, it would read your normal weight. This is because your weight is pushing down on the top of the scale, the floor is not letting the scale move, thereby compressing the springs within to a certain degree. Now, the elevator begins to moves upward. What happens? Your weight is still pressing down in the same way, but now the upwardly accelerating floor is pressing up more than it was. As a result, the displayed weight increases. At the top floor of the building, the unthinkable happens, the cable snaps and you plummet downward in gravitational freefall. In your last seconds, you look down at the scale to see it reads zero. You are still pushing down in the same fashion, but now the floor is accelerating downward at the same rate, meaning it is not pushing back, so the springs cannot compress, and the scale reads zero.

Now suppose you were in a small spacecraft with a bathroom scale far away from anything so that there is no gravity. You step on the scale and it reads zero. Now suppose the spacecraft fires its engines so that it accelerates upward. The pressing of the spacecraft's floor and your mass against the top and bottom of the scale compress the spring and your weight is given. If the spacecraft accelerated at 9.8 meters per second each second, what would the scale read? Your weight on Earth.

So, if you woke up to find yourself in a small room with nothing but a bathroom scale, could you figure out if you were in a non-accelerating reference frame in a gravitational field, in an accelerating spaceship in a gravity-free region, or some combination? No. No matter what the result, it could be equally well explained in either way. Not only that, but notice that if we tried to explain it as acceleration, then everything around you, no matter how much mass it has, would have to be held to have the same acceleration. But this is exactly what we want. Einstein could now argue that gravitation and acceleration were just different descriptions of the same thing.

But someone accelerating would "see" the light cones of events shifted at an angle compared to those at rest with respect to them. If acceleration and gravitation are equivalent descriptions, then gravitation should skew the light cones, but this is the same thing as warping space-time itself, that is, gravitation would bend the universe. This was the big insight. Einstein now just had to find the equations that described the geometry of the universe for any given distribution of matter and energy. To be successful, they had to satisfy two conditions – they had to be generally covariant, that is, keep the invariants the same regardless of the frame of reference, and they had to satisfy a criterion of uniqueness (Eindeutigkeit), that is, they had to give a unique value for the strength of the gravitational field at every point. The equations had to hold for all observers and for each one provide a fixed value for the field strength for each point in space-time.

Marcel Grossman once again coming to the rescue and helping Einstein acquire an understanding of the new tensor calculus, a more powerful mathematics needed for the project, he began to work. As he progressed, he reached a point where the project was promising. But then, he hit a snag. Einstein realized that he had field equations that were generally covariant, but they permitted him to do something strange. He could take a small region of space-time that was devoid of masses, he referred to such an empty region as a "hole," and twist the field values for the geometry of space in the hole in arbitrary ways. This meant that within the hole, the field equations would not uniquely nail down the physical values for the curvature of space-time. He could easily fix this, but it would cost him general covariance. If he wanted to save what Minkowski gave him in the extension of the theory, it seemed he would have to surrender the ability to give a complete and unique description of reality. But such a complete and unique description is the very point of a scientific theory. What to do?

Einstein struggled with this "hole problem" from late 1913 through 1915.[21] The original field equations seemed right, but they meant that he could not have both general covariance and uniqueness so they could not be right. He took the physics in several different directions, trying to avoid the problem. Ultimately, Einstein realized that it was not a problem of the physics, rather it was a philosophical problem; the trouble came not from the field equations, but from his interpretation of them. He was taking the values of the metric field, the curvature of space-time to be a real thing. But, after all, it was a hole, a region where there was nothing to be altered. It was a difference that made no difference. If he considered the different values in the hole to just be different descriptions of the same reality, the reality made up of coincidences, that is, events which lie at the intersection of world lines (in other words, actual interactions among things), then he would have a theory that described the world in a way that would work for all possible observers.

Einstein sets this out in what is called the "point-coincidence argument" in his 1916 paper "The Foundation of the General Theory of Relativity."

> The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).
>
> It is clear that a physical theory which satisfies this postulate will also be suitable for the general postulate of relativity…That this requirement of general co-variance takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time.
>
> The introduction of a system of reference serves no other

> purpose than to facilitate the description of the totality of such coincidences…[22]

What is real, Einstein argues is what is observationally verifiable, and what is verifiable are the interactions of things, the coincidences of world lines. That this line of reasoning, key to the understanding of the general theory of relativity was a claim about the nature of knowledge was not a fact that would escape the philosophical community for long.

Invariance and Uniqueness in Philosophy

When Einstein gave his first graduate seminar on the general theory of relativity at the University of Berlin in 1919, there were five attendees. One was Hans Reichenbach, a philosopher.

Reichenbach had been an engineer before writing a doctoral dissertation in the philosophy of physics, a field that did not exist at the time. His readers included a physicist who did not understand the philosophy and a philosopher who did not understand the mathematics or physics. When he had a chance as a newly minted Ph.D. to take Einstein's seminar, he jumped at the chance and the two would become friends and ultimately colleagues.

Reichenbach was quick to understand that the theory of relativity did not only overturn core beliefs in physics, but also in philosophy. The philosophical world at the time was dominated by the thought of Immanuel Kant. If one was not pursuing a neo-Kantian project of some sort, then one was most likely part of a conversation that was a response to Kant (say, Hegelian) or a response to a response to Kant (say, Nietzschean, Marxist, or existential). These reactions tended to take the conversation far afield when viewed from the perspective of a natural scientist, so those interested in the foundations of physics tended to remain within the Kantian conversation.

For Kant, the central question of philosophy was "How is synthetic a priori knowledge possible?", in other words, all observation must be a combination of sense data collected by the body with rational concepts provided by the mind. These mental notions must exist before the observations as they are preconditions for even the possibility of observing. We must take the raw sense data and put it through an intellectual filter to create the complex observations we have. Since these intellectual categories must come before observation, they cannot come from the world. But since they provide us with non-tautological knowledge, they cannot be merely logical, they must have content beyond their form. Where, then, could such knowledge come from?

Kant argues that it must preexist within the human mind; it must be innate, a standard part of the structure of the human consciousness. Included among these beliefs, Kant contends, are Euclidean geometry and Newtonian physics. They are the psychological rules by which we construct the world we see in all its complexity from the blur of colors our eyes feed into our minds. As such, they are not facts that can be confirmed or falsified by observation as they are the very rules by which all observations are created. Rather, they are undeniable truths about the structure of our minds.

Of course, both Euclidean geometry and Newtonian physics are denied by the general theory of relativity, a theory Einstein bases upon observation. In studying the theory with Einstein, Reichenbach realizes that Einstein has not only overturned the view we hold in terms of mathematics and physics, but in doing so Einstein completely undermines the philosophical foundation that was generally accepted for the basis of belief in the truth of these systems. In refuting Kant, we would now need a new philosophy to go with the new physical worldview. But rather than tie it to Einstein's theory – the way Kant was tied to Newton and therefore would require replacing when Einstein's theory found its successor – Reichenbach aimed to create a method that would be informed by our views of the world, but still allow us to understand how to go about forming and justifying them.

This would be the goal of Reichenbach's first book, *The Theory of Relativity and A Priori Knowledge*, dedicated to and vetted by Einstein, and written immediately following the seminar. In it, Reichenbach sketches out the fundamental axiomatic assertions of Kant's Newtonianism and those of the theory of relativity, showing how they are incompatible and how the observational consequences of the two diverge, providing support for those of Einstein. His introduction to the book begins:

> Einstein's theory of relativity has greatly affected the fundamental principles of epistemology. It will not serve any purpose to deny this fact or to pretend that the physical theory changed only the concepts of physics while the philosophical truths remained inviolate. Even though the theory of relativity concerns only relations of physical measurability and physical magnitudes, it must be admitted that these physical assertions contradict general philosophical principles. The philosophical axioms, even in their critical [viz., Kantian] form, were always formulated in such a way that they remained invariant with respect to specific interpretations but definitely excluded certain kinds of physical statements. Yet the theory of relativity selected exactly those statements that had been regarded as inadmissible and made them the guiding principles of its physical assumptions.[23]

Notice the way Reichenbach frames the discussion – "The philosophical axioms… were always formulated in such a way that they remained *invariant with respect to specific interpretation*." Here, we have the notion of invariance invoked yet again, but in an entirely new context. Where Cayley had used the notion to talk about the form of an equation under a transformation of coordinates and Einstein used the concept with respect to the reference frame of an observer, Reichenbach is now referring to the invariance of the meaning of sentences under different interpretations.

This is no accident. Reichenbach's new epistemological foundation inspired by and accounting for the theory of relativity will hinge on the notion of coordination. He makes clear that there is a difference between mathematical coordinates [Koordinaten] and his notion of coordinating statements [Zuordnung], but the words are both translated into English using the same notions of coordinating and it is clear that Reichenbach is playing metaphorically on the similarity. Indeed, for a coordination to be successful, he borrows the term uniqueness [Eindeutigkeit] that we saw in Einstein's point-coincidence argument in his 1916 paper.[24]

Further, the eventual philosophical stance that emerges from the work of Reichenbach and Moritz Schlick, a physicist trained under Max Planck turned philosopher who was the first to work on the philosophical ramifications of the theory of relativity[25] and who was a correspondent with Einstein, would find its central tenet in the point-coincidence argument. Logical Empiricism sought to eliminate meaningless propositions from philosophical discourse by use of a criterion of cognitive significance. This condition on the meaningfulness of propositions would take various forms as a principle of verifiability. Einstein had realized that the different formulations of the world with and without the hole were just different ways of describing the same reality. In the same way, Reichenbach and Schlick would argue that the meaning of a sentence is completely contained within the empirical content of the sentence and therefore that any two sentences or theories that had the same observable consequences were merely different ways of saying the same thing.[26]

The logical empiricists saw the job of philosophy to be quite different from its previous inception. Philosophical discourse, they contended had become mired down in meaningless chatter about pseudoquestions.[27] It had become impressive sounding nonsense. A question is a request for information. If there is such a fact to be found, then it would be determinable either through mere logical analysis of the language of the question in trivial cases or by checking the world, i.e., employing the means of science, in non-trivial cases. If there was no such information, then the question was not in fact a question, but nonsense, in other words, a pseudoquestion. Such it was, they claimed, with much of traditional and contemporary philosophy. The work of the philosopher, therefore, as they saw it was two fold: (i) to separate the meaningful from the meaningless, the real questions from the pseudoquestions, and ship them off to the physicist, biologist, psychologist, or whichever scientist was appropriate to find the relevant fact, and (ii) to provide the logical foundations justifying the methodologies of those scientific endeavors.

The deep attachment to science, mathematics, and logic amongst the logical empiricists came not only from an embrace of the advances in science from Einstein, mathematics from Klein, and logic from Bertrand Russell, but also from the times. In the shadow of the horrors of World War I with its trench warfare, chemical weapons, and mass death, the German speaking world became politically bifurcated with the pro-nationalist conservatives blaming the modernist, Jewish, and scientific trends for the ghastly results while the internationalist, pro-scientific left blamed the old religious, monarchic structures for their militarism, rigidity, and superstition.[28] The logical empiricists saw the scientific community as a functioning human community that fostered international cooperation and therefore a bulwark against war. Further, science would be able to undermine the over-simplistic and false worldviews that riled a population into such a frenzy that they could commit and condone the sorts of atrocities seen in Europe during the Great War.[29] Science and a worldview based on and modeled after science could save humanity from itself they thought.

Invariance in Gestalt Psychology

If the logical empiricists wanted to base all human belief and social structure on science, then they needed a clear and complete understanding of how science worked. An inextricable element of the scientific method is observation. Science provides evidence based upon the actual sense experiences of the scientist. They needed to understand the logical structure of inferences that took them from observations of particular things, like reading the number on a meter, to universal truths and laws of nature. But more basically, they also needed to understand how observation itself worked.

One of Kant's great insights, a realization that the logical empiricists cherished, is that the naïve view of early empiricists like John Locke who held that the mind is initially a *tabula rasa* written upon by observation could not possibly be correct. The world did not simply leave impressions on our minds. In *The Critique of Pure Reason*, Kant argues that there must be pre-existing mental structures to create the observations from the raw sense data fed in through our senses. Observation is a combination of the world sending messages through our sense organs and pre-established psychological means within our minds to order and make sense of sensation. If the logical empiricists were going to completely understand science, they would need to completely understand the science that explained how observation worked, observation that stood at the heart of science.

As it would happen, as Reichenbach was assembling a multi-disciplinary group of intellectuals to embark on this project through coordinated presentations, university seminars, and informal discussion groups, there was a group of psychologists at the University of Berlin who were working on exactly the question of interest.

Gestalt psychology began in the late 1910s with Max Wertheimer's work in Frankfurt where he took two young students under his wing, Wolfgang Köhler and Kurt Koffka. The three would be the founding figures in the Gestalt movement.

Wertheimer and Köhler took positions at Berlin after the war and while pursuing work that would establish Gestalt doctrines as major advances in the history of psychology, they also became active members in Reichenbach's discussion groups. They were well aware of the philosophical import of their work. In his address, "Über Gestalttheorie" to the Kant Society in Berlin 1924, Wertheimer makes explicit that Gestalt psychology is part of the larger movement of the times,

> What is Gestalt theory and what does it intend? Gestalt theory was the outcome of concrete investigations in psychology, logic, and epistemology.[30]

The terms "logic" and "epistemology," when used in Berlin in 1924 would clearly point to Reichenbach.

The links between Reichenbach's philosophical circle and the Gestalt community were deep. Reichenbach was deeply impressed by Wertheimer's works. Even though Reichenbach had to flee Germany once the universities were purged in 1933 by Hitler, taking a position for five years in Istanbul then ultimately ending up at UCLA, Wertheimer's books remained part of Reichenbach's personal library until his death.

Kurt Lewin, widely held to be the founder of social psychology, was a contemporary of Reichenbach who would work on his *Habitationschrift* at Berlin when Reichenbach was there taking Einstein's seminar. The two would collaborate on work and Lewin was the only person other than Einstein whom Reichenbach would show drafts of his work in the foundations of physics.[31] Lewin, in addition to Reichenbach's student Kurt Grelling[32] would work in both the fields of philosophy of science and Gestalt psychology in collaboration with Wertheimer, Köhler, and Koffka.

Köhler was widely celebrated as a teacher. His lectures would overflow their venues. Yet, in spite of his renown and his own duties, at Reichenbach's behest, it was Köhler who took over supervision of Reichenbach's doctoral students, most famously Carl Gustav Hempel, after Reichenbach's forced departure from Berlin.

The heart of the Gestalt theory was the Kantian idea that there were preexisting structures in the mind needed to make sense of raw perception. What made Gestalt theory unique was the claim that this only happened when the mind interpreted input as a whole.

> Is it really true that when I hear a melody I have a *sum* of individual tones (pieces) which constitute the primary foundation of my experience? Is not perhaps the reverse of this true? What I really have, what I hear of each individual note, what I experience in each place in the melody is a *part* which is itself determined by the character of the whole. What is given me by the melody does not arise (through the agency of any auxiliary factor) as a *secondary* process from the sum of the pieces as such. Instead, what takes place in each single part already depends upon what the whole is. The flesh and blood of a tone depends from the start upon its role in the melody: a *b* as a leading tone to cis something radically different from the b as a tonic. It belongs to the flesh and blood of the things given in experience, how, in what role, in what function they are in the whole.[33]

The mind does not create the whole from the part, but rather understands the part only in terms of the whole, often supplying parts of the whole that may not actually be experienced as in the famous Gestalt examples of incomplete shapes clearly seen.

When the Gestalt theorists write of this relation of part to whole, what term do we find employed? "Invariance," of course.

> *The Invariants.* Therefore we shall discard the empiristic theory as an ultimate explanation of our framework, without, however, raising the claim that experience can have no effect at all upon it. Such a claim would, in the present state of our knowledge, be unwarranted. Having rid ourselves of the empiristic bias we find in our last examples a very simple principle: such parts of the behavioral environment as become part of our general spatial framework assume one of the main spatial directions. Let us see what this principle means in our examples. When we look through the window of our mountain-railway carriage, this window becomes our spatial framework and appears, therefore, in normal, horizontal-vertical orientation. The contours of the objects seen through the window

> do not intersect with the sash at right angles. Therefore, if the sash is seen as horizontal, these objects cannot be seen as vertical, but must appear leaning away from us on the ascent, and towards us on the descent. If Fig. 72 gives a somewhat exaggerated picture of the real positions of the window and the telegraph pole, then it shows at the same time why the telegraph pole cannot appear vertical when the window becomes the framework and thereby horizontally-vertically oriented. All one has to do is turn this picture until the lower side of the window is horizontal; then of course the telegraph pole is tilted to the right as much as in our drawing the window is tilted to the left. The angle between the pole and the window sash, then, determines the relative localization of the two objects with regard to each other, whereas their absolute localization is determined by those parts of the field which form the spatial framework. If one sticks one's head out of the window, the telegraph pole will soon look vertical; when then, without losing sight of it, one withdraws the head, the telegraph pole will still look vertical and the windows, the whole carriage, tilted. One factor in these two situations is invariant, the angle between ground and object…
>
> We shall find the same principle, involving naturally other invariants, operative in the field of colour and of movement as well: relative properties of the stimulus distribution determining relative properties of the objects and events in the behavioural world, but the absolute properties of these latter depending upon a new factor, which in our case of the spatial framework is the stress of this framework towards the main directions of space.[34]

Koffka's use of the notion of invariant as that which is real and establishes a framework for truth beyond the relative is more than a little reminiscent of the point-coincidence argument from Einstein and the notion of uniqueness in interpretation from Reichenbach.

Koffka is careful to draw a distinction between the image on the retina and the phenomenological or behavioural observation. We can have the same image on our retina at different times and yet see different things. As he points out in the passage above, the way the behavioural experience is created by a combination of the sense organs and the mind will create relative truths, that is, truths that are experienced relative to relative factors. But then there are invariant factors and these create experiential truths that outrun particular frames of reference. This is precisely the sort of invariance of interpretation suggested by Einstein's 1916 theory of general relativity and appropriated by Reichenbach as the central feature of his epistemology.

Through the relationships, both intellectually and personally, between the algebra of Cayley, the geometry of Klein, the physics of Einstein, the interpretation of Minkowski, the philosophy of Reichenbach, and the psychology of Wertheimer, Köhler, and Koffka, we see the term "invariance" assume covariant senses in several intellectual frames of reference. Yet, through it all, there is an invariance to "invariance," a sense of the reality of the absolute, but an absolute that appears and can only be completely understood in the particularity of its circumstances. Science is a human endeavor and as in all other human occupations who you know will influence what you know and what you do.

As Newton so rightly pointed out, standing on the shoulders of giants does not diminish what one sees from their wonderful vantage point. One fascinating aspect of the story of science, though, is that the pyramid of giants whose shoulders one stands upon gets very wide at the base, thereby including giants one might not expect.

Endnotes

1. Crilly, 2006, p. 120 and Bell, 1937, p. 382.
2. Bell, 1937, p. 379.
3. Jaroma and Gimbel, 2003.
4. Sylvester, 1888, p. 152.
5. Crilly, 2006, p. 87.
6. Reid, 1970, p. 19.
7. Ibid.
8 From William Henry Young's biography of Klein, quoted in Kramer, p. 428.
9. Yaglom, p. 189, fn. 181.
10. From William Henry Young's biography of Klein, quoted in Kramer, p. 430.
11. Reid, 1970, p. 6.
12. Quoted in Reid, 1970, p. 16.
13. Einstein, 1949, p. 15.
14. Einstein, 1905, p. 37.
15. Ibid., 38.
16. Quoted in Reid, 1970, p. 105.
17. Quoted in Reid, 1970, p. 112.
18. Einstein, 1905, p. 59.
19. Minkowski, 1908, p. 75.
20. Einstein, 1916*b*, p. 61.
21. For a detailed account of Einstein and the hole problem, see Stachel 1980.
22. Einstein, 1916*a*, p. 117.
23. Reichenbach, 1920, pp. 1-2.
24. For a detailed discussion of the relation between the Point-Coincidence argument and the development of 20th century analytic philosophy, see Ryckman, 1992.
25. Schlick, 1919.
26. See Reichenbach, 1926, and Schlick, 1919.
27. See Carnap, 1932, and Ayer, 1952, for a couple of the clearest and most aggressive versions of this line.
28. See Stern, 1999.
29. See Gimbel, 2002.
30. Wertheimer, 1924, p. 1.
31. A detailed account of the Lewin/Reichenbach relationship is detailed in Padovani, 2008.

32. A biographical sketch of the life and tragic death of Kurt Grelling can be found in Luchins and Luchins, 2000.
33. Wertheimer, 1924, p. 5.
34. Koffka, 1935, pp. 218-219.

Bibliography

Ayer, A.J. 1952. *Language, Truth, and Logic.* (New York: Dover).

Bell, E.T. 1937. *Men of Mathematics.* (New York: Simon and Schuster).

Carnap, Rudolf. 1932. "The Elimination of Metaphysics Through the Logical Analysis of Language." *Logical Positivism.* (ed.) A.J. Ayer. (Glencoe: Free Press, 1959).

Crilly, Tony. 2006. *Arthur Cayley: Mathematician Laureate of the Victorian Age.* (Baltimore: Johns Hopkins University Press).

Einstein, Albert. 1905. "On the Electrodynamics of Moving Bodies," *The Principle of Relativity.* (New York: Dover, 1952). 35-65.

Einstein, Albert. 1916a. "The Foundation of the General Theory of Relativity," *The Principle of Relativity.* (New York: Dover, 1952). 109-164.

Einstein, Albert. 1916b. *Relativity: The Special and the General Theory.* (New York: Wings, 1961).

Einstein, Albert. 1949. "Intellectual Autobiography." In *Albert Einstein: Philosopher-Scientist.* (ed.) Paul Schillp. (La Salle: Open Court).

Gimbel, Steven. 2002. "Politics and the Wissenschaftliche Weltanschauung: Neurath and Reichenbach on Science and Socialism." *Siblings Under the Skin: Feminism, Social Justice, and Analytic Philosophy.* (ed.) Shayn Clough. (Aurora: Davies Publishing Group).

Jaroma, John and Gimbel, Steven. 2003. Sylvester: Ushering in the Modern Era of Research on Odd Perfect Numbers. *Integers: Electronic Journal of Combinatorial Number Theory.* Volume 3.

Koffka, Kurt. 1935. *Principles of Gestalt Psychology.* (New York: Harbinger, 1963).

Kramer, Edna. 1970. *The Nature and Growth of Modern Mathematics.* (Princeton: Princeton University Press).

Luchins, Abraham and Luchins, Edith. 2000. "Kurt Grelling: Steadfast Scholar in a Time of Madness." *Gestalt Theory.* Volume 22, Number 4. pp. 228-281.

Minkowski, Hermann. 1908. "Space and Time," *The Principle of Relativity.* (New York: Dover, 1952). 73-91.

Padovani, Flavia. 2008. *Probability and Causality in the Early Works of Hans Reichenbach.* University of Geneva. Unpublished dissertation.

Reichenbach, Hans. 1920. *Theory of Relativity and A Priori Knowledge.* (Berkeley: University of California Press, 1969).

Reichenbach, Hans. 1926. *Philosophy of Space and Time.* (New York: Dover, 1958).

Reid, Constance. 1970. *Hilbert.* (New York: Springer Verlag, 1970).

Ryckman, T.A. 1992. "'P(oint)-C(oincident) Thinking': The Ironical Attachment of Logical Empiricism to General Relativity (and Some Lingering Consequences)." *Studies in the History and Philosophy of Science.* Volume 23, Number 3., pp. 471-497.

Schlick, Moritz. 1919. Space and Time in Contemporary Physics: *An Introduction to the*

Theory of Relativity and Gravitation. (New York: Dover, 1963).
Schlick, Moritz. 1925. *General Theory of Knowledge*. (La Salle: Open Court, 1974).
Stachel, John. 1980. "Einstein's Search for General Covariance, 1912-1915." In *Einstein and the History of General Relativity*. (eds.) Howard, Don and John Stachel. (Boston: Birkhäuser). 63-100.
Stern, Fritz. 1999. *Einstein's German World*. (Princeton: Princeton University Press).
Sylvester, J.J. 1888. "Note on a Proposed Addition to the Vocabulary of Ordinary Arithmetic." *Nature*. Volume 37. pp. 152-153.
Wertheimer, Max. 1924. "Gestalt Theory" *A Sourcebook of Gestalt Psychology*. (ed.) Willis D. Ellis. (New York: Humanities Press, 1955).
Yaglom, I.M. 1988. *Felix Klein and Sophus Lie*. (Boston: Birkhäuser).