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# Solving the Debt Crisis on Graphs - Solutions

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# Solving the Debt Crisis on Graphs - Solutions

**Abstract**
We begin by noting that solutions to these puzzles are not unique. In particular, doing the `lending' action from each of the vertices once brings us back to where we started. Moreover, the act of doing the `borrowing' action from one vertex is equivalent to doing the`lending' action from each of the other vertices. In particular, without loss of generality one can assume that there is (at least) one vertex for which you do neither action and for all other vertices you do the `lending' action a nonnegative number of times. Below we give possible solutions to four of the puzzles by showing the number of times one lends from each vertex in order to eliminate all debt.

**Keywords**
debt crisis, puzzle

**Disciplines**
Mathematics

# Solving the Debt Crisis on Graphs

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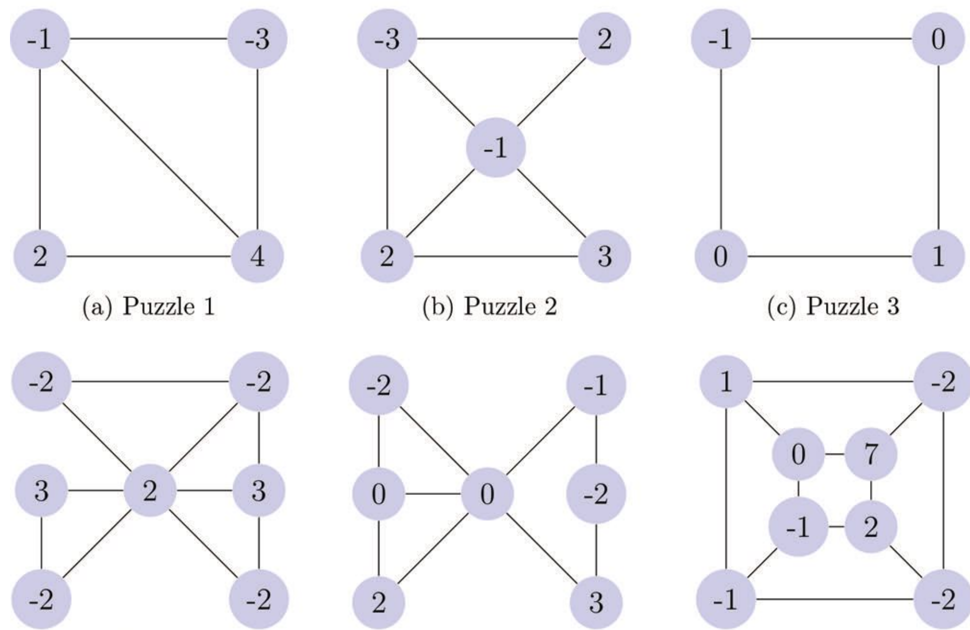
Puzzle Page

These puzzles are set on a graph in which each node is labeled with an integer. Think of the nodes as people and the integers as an amount of money that the person has in the bank (or is in debt). At each turn, you choose a node and either give one dollar from it to each of its neighbors or take a dollar from each of its neighbors. Moves preserve the total number of dollars in the system, so if the number is initially non-negative, then it might be possible to do a sequence of moves that will eventually lead to none of the nodes being "in debt." For each of the following examples, try to either find a sequence of moves that will lead to none of the nodes being in debt or explain why no such sequence exists.

In general, it is difficult to tell when there exists a winning strategy for a given puzzle.

A theorem due to Baker and Norine, proven in their paper "Riemann-Roch and Abel-Jacobi Theory on a Finite Graph" (arxiv.org/abs/math/0608360), puts these puzzles in the more general context of Chip-Firing games and gives a partial answer to this question.

**Theorem 0.1** Assume we are given a puzzle with $E$ edges, $N$ nodes, and $D$ total dollars in the system. If $D > E - N$, then there is a winning strategy. Moreover, for each $0 \leq D \leq E - N$ there exist puzzles with winning strategies as well as puzzles with no winning strategy.




(a) Puzzle 1 (b) Puzzle 2 (c) Puzzle 3

(d) Puzzle 4 (e) Puzzle 5 (f) Puzzle 6

Knowing this theorem gives one a quick way to come up with puzzles that can keep you occupied on long airplane flights or at particularly boring committee meetings: Draw a graph with $N$ nodes and $E$ edges and an initial configuration with a total of $E - N + 1$ dollars shared between the nodes, and try to find a winning strategy. It is also interesting to try to come up with examples with $E - N$ dollars for which no winning strategy exists! The answer is online at maa.org/pubs/FOCUS Aug-Sep12_puzzles.html and will be in the October/November issue.

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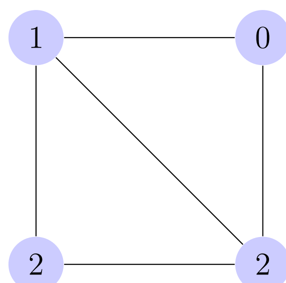
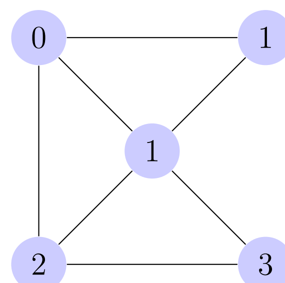
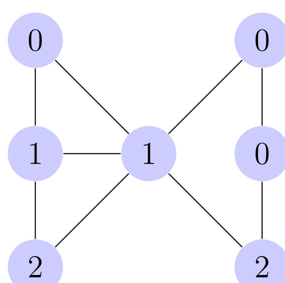
We begin by noting that solutions to these puzzles are not unique. In particular, doing the ‘lending’ action from each of the vertices once brings us back to where we started. Moreover, the act of doing the ‘borrowing’ action from one vertex is equivalent to doing the ‘lending’ action from each of the other vertices. In particular, without loss of generality one can assume that there is (at least) one vertex for which you do neither action and for all other vertices you do the ‘lending’ action a nonnegative number of times. Below we give possible solutions to four of the puzzles by showing the number of times one lends from each vertex in order to eliminate all debt.



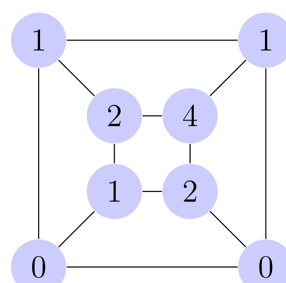

(a) Puzzle 1




(b) Puzzle 2




(c) Puzzle 5




(d) Puzzle 6

The third and fourth puzzles have no solutions. In both cases, we note that the system has zero net dollars in it, so in order for no node to be in debt we must move to a position where each node has exactly zero dollars. In the third puzzle, we further note that this implies that one must ‘lend’ from the upper-right corner and lower-left corner the same number of times due to symmetry. If one does this, then the number of dollars on the other two nodes will be odd and therefore cannot be zero. To see that the fourth puzzle has no solution, we focus our attention on the two nodes in the lower left-hand corner of the graph.

In particular, assume that $x$ is the number of dollars on the center-left node and $y$ is the number of dollars on the lower-left node, so that initially $x = 3$, $y = -2$, and the quantity $x - y = 5$. One can see that any legal move will change the quantity $x - y$ by a multiple of 3, and in particular we can never obtain $x - y = 0$. This shows there is no way to get this graph out of debt.