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Calculus I Companion

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Calculus I Companion

Description

A course pack for supplementing Calculus 1 with algebra, geometry, trigonometry, and precalculus topics, including reading material, activities, and practice problems assembled from various OER texts.

Keywords

calculus prerequisite material, just-in-time precalculus, active learning, OER, open textbook

Disciplines

Mathematics

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MATH 110
CALCULUS I COMPANION

Math 110

Calculus 1 Companion

eds. Beth Campbell Hetrick, Kim Spayd
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The following pages are all compiled from these open educational resources:

- Abramson, J., *Algebra and Trigonometry 2e*, OpenStax, 2021,
<https://openstax.org/details/books/algebra-and-trigonometry-2e>
- Abramson, J., *Precalculus 2e*, OpenStax, 2021,
<https://openstax.org/details/books/precalculus-2e>
- Abramson, J. and S. North, *College Algebra with Corequisite Support 2e*, OpenStax, 2021,
<https://openstax.org/details/books/college-algebra-corequisite-support-2e>
- Boelkins, M., *Active Prelude to Calculus*, 2019,
<https://activecalculus.org/prelude/book-1.html>
- Boelkins, M., et al. *Active Calculus*, 2023,
<https://activecalculus.org/single/book-1.html>
- Clontz, S. and D. Lewis, eds., *Calculus for Team-Based Inquiry Learning*, 2022
<https://teambasedinquirylearning.github.io/calculus/2023e/cal1.html>
- Yoshiwara, K., *Modeling, Function, and Graphs*, 2018,
<https://yoshiwarabooks.org/mfg/MFG.html>

This course pack is available at <https://cupola.gettysburg.edu/>



Colophon

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Sections 3.2.2 and 4.5 come from *Modeling, Functions, and Graphs*: © 2018 Katherine Yoshiwara under the GNU Free Documentation License.

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1 Metacognition

Calculus Fun Fact: The roots of calculus lie in some of the oldest geometry problems on record. The Egyptian Rhind papyrus (c. 1650 bce) gives rules for finding the area of a circle and the volume of a truncated pyramid.

COREQUISITE SKILLS

Learning Objectives

- Identify the study skills leading to success in a college level mathematics course.
- Reflect on your past math experiences and create a plan for improvement.

Objective 1: Identify the study skills leading to success in a college level mathematics course.

Welcome to your algebra course! This course will be challenging so now is the time to set up a plan for success. In this first chapter we will focus on important strategies for success including: math study skills, time management, note taking skills, smart test taking strategies, and the idea of a growth mindset. Each of these ideas will help you to be successful in your college level math course whether you are enrolled in a face-to-face traditional section or an online section virtual section.

Complete the following survey by checking a column for each behavior based on the frequency that you engage in the behavior during your last academic term.

Behavior or belief:	Always	Sometimes	Never
1. Arrive or log in early to class each session.			
2. Stay engaged for the entire class session or online meeting.			
3. Contact a fellow student and my instructor if I must miss class for notes or important announcements.			
4. Read through my class notes before beginning my homework.			
5. Connect with a study partner either virtually or in class.			
6. Keep my phone put away during classes to avoid distractions.			
7. Spend time on homework each day.			
8. Begin to review for exams a week prior to exam.			
9. Create a practice test and take it before an exam.			
10. Find my instructor's office hours and stop in either face-to-face or virtually for help.			
11. Locate the math tutoring resources (on campus or virtually) for students and make note of available hours.			
12. Visit math tutoring services for assistance on a regular basis (virtual or face-to-face).			
13. Spend at least two hours studying outside of class for each hour in class (virtual or face-to-face).			
14. Check my progress in my math course through my college's learning management system.			
15. Scan through my entire test before beginning and start off working on a problem I am confident in solving.			

Behavior or belief:	Always	Sometimes	Never
16. Gain access to my math courseware by the end of first week of classes.			
17. Send an email to my instructor when I need assistance.			
18. Create a schedule for each week including time in class, at work and study time.			
19. Read through my textbook on the section we are covering before I come to class or begin virtual sessions.			
20. Feel confident when I start a math exam.			
21. Keep a separate notebook for each class I am taking. Divide math notebooks or binders into separate sections for homework, PowerPoint slides, and notes.			
22. Talk honestly about classes with a friend or family member on a regular basis.			
23. Add test dates to a calendar at the beginning of the semester.			
24. Take notes each math class session.			
25. Ask my instructor questions in class (face-to-face or virtual) if I don't understand.			
26. Complete nightly homework assignments.			
27. Engage in class discussions.(virtual or face-to-face)			
28. Recopy my class notes more neatly after class.			
29. Have a quiet, organized place to study.			
30. Avoid calls or texts from friends when I'm studying.			
31. Set study goals for myself each week.			
32. Think about my academic major and future occupation.			
33. Take responsibility for my study plan.			
34. Try different approaches to solve when I get stuck on a problem.			
35. Believe that I can be successful in any college math course.			
36. Search for instructional videos online when I get really stuck on a section or an exercise.			
37. Create flashcards to help in memorizing important formulas and strategies.			

Behavior or belief:	Always	Sometimes	Never
Total number in each column:			
Scoring:	Always: 4 points each	Sometimes: 2 points each	Never: 0 points each
Total Points:			0

Practice Makes Perfect

Identify the study skills leading to success in a college level mathematics course.

- Each of the behaviors or attitudes listed in the table above are associated with success in college mathematics. This means that students who use these strategies or are open to these beliefs are successful learners. Share your total score with your study group in class and be supportive of your fellow students!
- Based on this survey create a list of the top 5 strategies that you currently utilize, and feel are most helpful to you.
 -
 -
 -
 -
 -
- Based on this survey create a list of the top 5 strategies that interest you, and that you feel could be most helpful to you this term. Plan on implementing these strategies.
 -
 -
 -
 -
 -

Objective 2: Reflect on your past math experiences and create a plan for improvement.

- It's important to take the opportunity to reflect on your past experiences in math classes as you begin a new term. We can learn a lot from these reflections and thus work toward developing a strategy for improvement. In the table below list 5 challenges you had in past math courses and list a possible solution that you could try this semester.

Challenge	Possible solution
1.	
2.	
3.	
4.	
5.	

- Write your math autobiography. Tell your math story by describing your past experiences as a learner of mathematics. Share how your attitudes have changed about math over the years if they have. Perhaps include what you love, hate, dread, appreciate, fear, look forward to, or find beauty in. This will help your teacher to better

understand you and your current feelings about the discipline.

3. Share your autobiographies with your study group members. This helps to create a community in the classroom when common themes emerge.

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattle herders, and traders used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century CE in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century CE, negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, \dots\}$ where the ellipsis (...) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\begin{array}{ccc} \text{negative integers} & \text{zero} & \text{positive integers} \\ \dots, -3, -2, -1, & 0, & 1, 2, 3, \dots \end{array}$$

The set of **rational numbers** is written as $\{\frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

- (a) a terminating decimal: $\frac{15}{8} = 1.875$, or (b) a repeating decimal: $\frac{4}{11} = 0.36363636 \dots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

EXAMPLE 1

Writing Integers as Rational Numbers

Write each of the following as a rational number.

- (a) 7 (b) 0 (c) -8

Solution

Write a fraction with the integer in the numerator and 1 in the denominator.

- (a) $7 = \frac{7}{1}$ (b) $0 = \frac{0}{1}$ (c) $-8 = -\frac{8}{1}$

1.2 Exponents and Scientific Notation

Learning Objectives

In this section, you will:

- › Use the product rule of exponents.
- › Use the quotient rule of exponents.
- › Use the power rule of exponents.
- › Use the zero exponent rule of exponents.
- › Use the negative rule of exponents.
- › Find the power of a product and a quotient.
- › Simplify exponential expressions.
- › Use scientific notation.

COREQUISITE SKILLS

Learning Objective:

- › Plan your weekly academic schedule for the term.

Objective 1: Plan your weekly academic schedule for the term.

1. Most college instructors advocate studying at least 2 hours for each hour in class. With this recommendation in mind, complete the following table showing credit hours enrolled in, the study time required, and total time to be devoted to college work. Assume 2 hours of study time for each hour in class to complete this table, and after your first exam you can fine tune this estimate based on your performance.

Credit hours (hours in class)	Study time outside of class	Total time spent in class and studying
9	(2,2)	(2,3)
12	(3,2)	(3,3)
15	(3,2)	(3,3)
18	(4,2)	(4,3)
21	(5,2)	(5,3)

Consider spending at least 2 hours of your study time each week at your campus (or virtual) math tutoring center or with a study group, the time will be well spent!

2. Another way to optimize your class and study time is to have a plan for efficiency, meaning make every minute count. Below is a list of good practices, check off those you feel you could utilize this term.

Best practices:	Will consider:	Not for me:
1. Attend each class session. It will take much more time to teach yourself the content.		
2. Ask your instructor. If you are unsure of a concept being taught in class, ask for clarification right away. Your instructor is an expert in their field and can provide the most efficient path to understanding.		

Best practices:	Will consider:	Not for me:
<p>3. Be prepared for each class. Having completed prior assignments can go a long way in math understanding since mastery of most learning objectives depends on knowledge of prior concepts. Also, reading through a section prior to class will help to make concepts much clearer.</p>		
<p>4. Stay organized. Keeping your math materials in a 3-ring binder organized by lecture notes, class handouts, PowerPoint slides, and homework problems will save you time in finding materials when you need them. Having two spiral notebooks dedicated to math works well too, use one for class notes and one for homework assignments.</p>		
<p>5. Find a study partner. Making a connection either in class or virtually with a fellow student can save time in that now there are two sources for gathering important information. If you have to miss class or an online session for an important appointment, your study partner can provide you class notes, share in-class handouts, or relay announcements for your instructor. Study partner's name: Study partner's number: Study partner's email address:</p>		
<p>6. Begin exam review time by reworking each of the examples your instructor worked in class. Your instructor will emphasize the same topics in both lecture and on exams based on student learning objectives required by your college or university or even the state where the course is offered. Follow their lead in assigning importance to an objective and master these topics first.</p>		

3. Creating your Semester Calendar- complete the following weekly schedule being sure to label
- time in classes
 - study time for classes
 - time at work.

Optional: also include if you want a more comprehensive view of your time commitments

- time spent exercising
- time with family and friends.

Term: _____

Name: _____

Date: _____

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
6:30-7:00am							
7:00-7:30am							
7:30-8am							
8-8:30am							

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
8:30-9am							
9-9:30am							
9:30-10am							
10-10:30am							
10:30-11am							
11-11:30am							
11:30-12pm							
12-12:30pm							
12:30-1pm							
1-1:30pm							
1:30-2pm							
2-2:30pm							
2:30-3pm							
3-3:30pm							
3:30-4pm							
4-4:30pm							
4:30-5pm							
5-5:30pm							
5:30-6pm							
6-6:30pm							
6:30-7pm							
7-7:30pm							
7:30-8pm							
8-8:30pm							
8:30-9pm							

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
9-9:30pm							
9:30-10pm							
10-10:30pm							
10:30-11pm							
11-11:30pm							

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, x , but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \cdot x^4 &= \overset{3 \text{ factors}}{x \cdot x \cdot x} \cdot \overset{4 \text{ factors}}{x \cdot x \cdot x \cdot x} \\ &= \overset{7 \text{ factors}}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents*.

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n , the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

Real-World Applications

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90,000 + 160,000}$. What is the length of the guy wire?
66. A car accelerates at a rate of $6 - \frac{\sqrt{4}}{\sqrt{t}}$ m/s² where t is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

67. $\frac{\sqrt{8}-\sqrt{16}}{4-\sqrt{2}} - 2\frac{1}{2}$

68. $\frac{4\frac{3}{2}-16\frac{3}{2}}{8\frac{1}{3}}$

69. $\frac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}} \cdot \frac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

70. $\frac{a}{a-\sqrt{c}}$

71. $\frac{x\sqrt{64y}+4\sqrt{y}}{\sqrt{128y}}$

72. $\left(\frac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\frac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\frac{\sqrt[3]{64+4\sqrt{256}}}{\sqrt{64+4\sqrt{256}}}}$

1.4 Polynomials

Learning Objectives

In this section, you will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

COREQUISITE SKILLS

Learning Objectives

- Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Objective 1: Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Stanford University psychologist and researcher, Carol Dweck, PH.D., published a book in 2006 called *"Mindset, The New Psychology of Success"*, which changed how many people think about their talents and abilities. Based on decades of research Dr. Dweck outlined two mindsets and their influence on our learning.

Dr. Dweck's research found that people who believe that their abilities could change through learning and practice (**growth mindset**) more readily accepted learning challenges and persisted through these challenges. While individuals who believe that knowledge and abilities come from natural talent and cannot be changed (**fixed mindset**) more often become discouraged by failure and do not persist.

Her research shows that if we believe we can learn and master something new, this belief greatly improves our ability to

learn.

1. Read through the following illustration based on Dr. Dweck's work.
2. It's important to note that we as individuals do not have a strict fixed or growth mindset at all times. We can lean one way or another in certain situations or when working in different disciplines or areas. For example, a person who often plays video games may feel they can learn any new game that is released and be confident in these abilities, but at the same time avoid sports and are fixed on the idea that they will never excel at physical activities. In terms of learning new skills in mathematics, which mindset, growth or fixed, best describes your beliefs as of today? Explain.

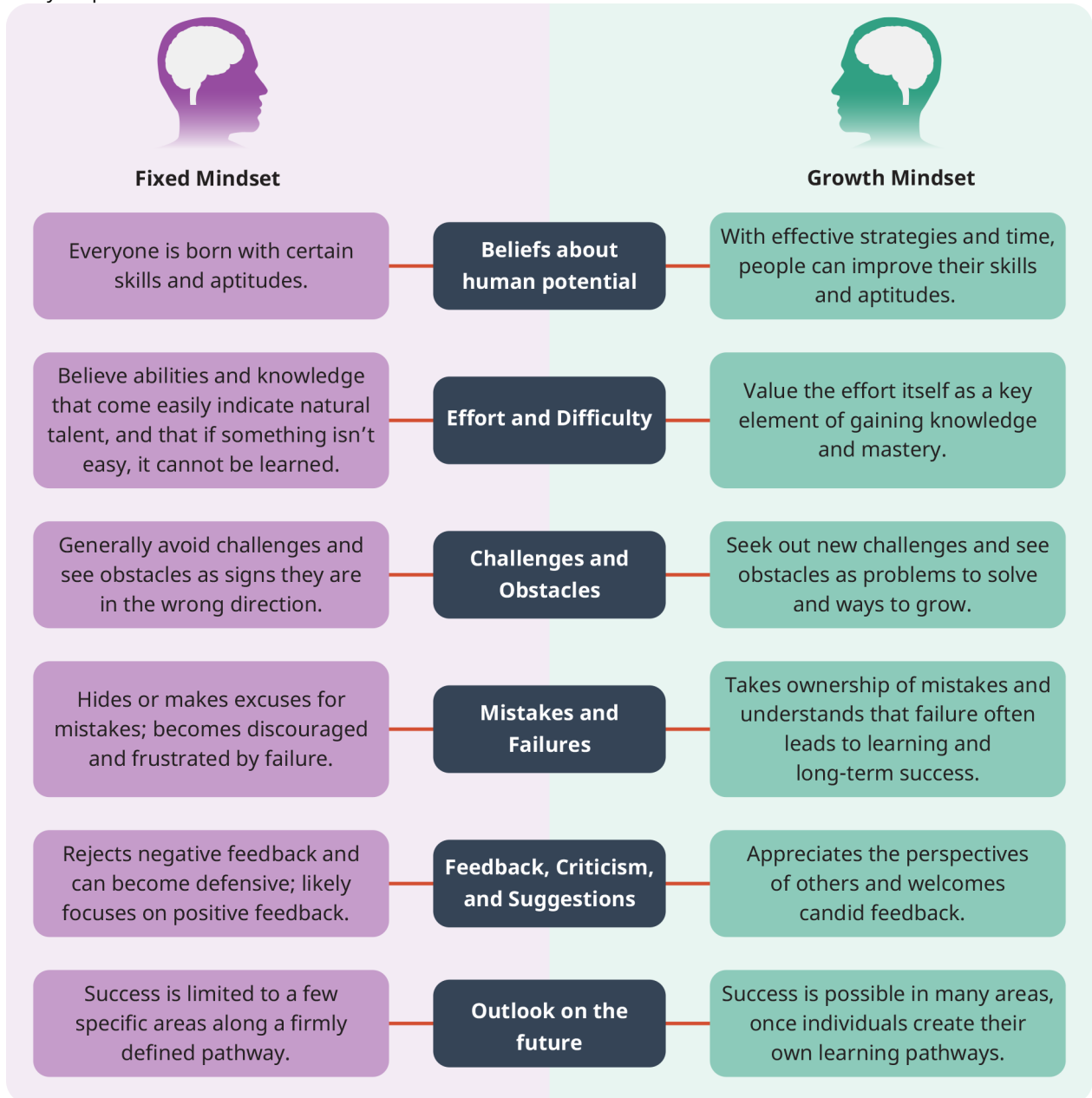


Figure 1 The differences between fixed and growth mindset are clear when aligned to key elements of learning and personality. (Credit: Adapted for OpenStax *College Success*, based on work by Dr. Carol Dweck)

3. Identify each of the following statements as coming from a student with a fixed mindset or with a growth mindset.

Statement	Fixed or Growth Mindset?
a. I've never been good at math, so I'll be happy just getting a D in this course.	
b. I hear that this instructor is really great, I'm excited to start this new term.	
c. I need to try harder in this class and put in more study time. I have the rest of the term to improve my performance.	
d. I hate math.	
e. This activity is dumb, I don't think it will help me.	
f. That exam was tough, but I'm going to rework it during my study time and get these concepts down before my final.	
g. I'm up for the challenge of this course.	
h. Some people are just better in math than me.	
i. Intelligence is something you have to work for.	
j. I find it best to erase every mistake I make in my homework and try to forget about it.	
k. I try to learn from my mistakes and make note of them.	
l. I'm not going to raise my hand to answer this question in class. I'll just be wrong.	
<p>4. Mindsets can be changed. As Dr. Dweck would say "You have a choice. Mindsets are just beliefs. They are powerful beliefs, but they are something in your mind and you can change your mind." Think about what you would like to achieve in your classes this term and how a growth mindset can help you reach these goals. Write three goals for yourself below.</p> <ol style="list-style-type: none"> 1. 2. 3. 	

Maahi is building a little free library (a small house-shaped book repository), whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which people can take and donate books. Maahi wants to find the area of the front of the library so that they can purchase the correct amount of paint. Using the measurements of the front of the house, shown in [Figure 1](#), we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

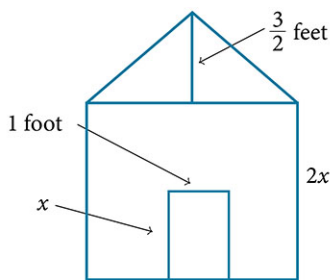


Figure 1

First find the area of the square in square feet.

Exam Preparation Strategies	Always	Sometimes	Never
1. Rework each of the examples my instructor did in class.			
2. Create note cards to help in memorizing important formulas and problem-solving strategies for the exam.			
3. Create a study schedule for each math exam and begin to study for the exam at least one week prior to the date. Spaced practice over 5-7 days is much more effective than cramming material in 1-2 sessions.			
4. Work the review exercises at the end of each chapter of the text.			
5. Visit my instructor's office hours when I need assistance in preparing for an exam.			
6. Spend time on note interactions (see the section on Cornell notes) each day.			
7. Create a practice test using the questions I identified in my class notes (see the section on Cornell notes) and take it the week before the exam.			
8. Review each of the student learning objectives at the beginning of all sections covered on the exam and use this list as a checklist for exam preparation.			
9. Ask your instructor how many questions will be on the exam and if they award partial credit for work shown.			
10. Work through the practice test at the end of each chapter of the text.			
11. Get a good night's sleep the night before my exam.			
12. Come to each exam prepared with a goal of earning an A.			

Exam Day Behaviors and Strategies	Always	Sometimes	Never
13. Make sure to grab a healthy breakfast the day of the exam.			
14. Arrive or log in early to class on exam days.			
15. Keep my phone put away in my bag during exams to avoid distractions.			
16. Try to relax and take a few deep breaths before beginning the exam.			
17. Use a pencil so that I can make corrections neatly.			
18. Read through all directions before beginning the exam.			
19. Write formulas that are memorized in the margins, top or back of the test to reference when needed.			

Exam Day Behaviors and Strategies	Always	Sometimes	Never
20. Scan through my entire test before beginning and start off working on a problem I am confident in solving.			
21. Work each of the questions that I find easier first.			
22. Keep track of time. Do a quick assessment of how much time should be spent on each question.			
23. Try different approaches to solve when I get stuck on a problem.			
24. Draw a diagram when solving an application problem.			
25. Do some work on each question.			
26. Work neatly and show all steps.			
27. Make sure to attach units to final answers when units are given in the problem. (for example: cm, \$, or feet/second)			
28. Stay working for the entire class session or online exam session. If finished early I use the additional time to review my work and check answers.			
29. Circle final answers or write each on the answer blank.			

After the Exam Behaviors and Strategies	Always	Sometimes	Never
30. I work back through my exams after they are returned, writing corrections in another color or highlighting them for future reference.			
31. Keep my old exams in a binder or notebook and use this assessment to review for my final exam.			
32. Take responsibility for my exam performance and try to learn from the experience.			
33. Reflect on the test taking experience and make a list for yourself on what to do differently next time.			
34. Reflect on your feelings while taking the exam. Plan to replace any negative self-statements with positive ones on future exams.			
35. Celebrate my success after doing well on an exam! Talk to a friend or family member about my progress.			

Scoring			
Total Number in Each Column			
Scoring:	Always: 4 points each	Sometimes: 2 points each	Never: 0 points each
Total points:			0

Practice Makes Perfect

Practice: Identify the skills leading to successful preparation for a college level mathematics exam.

- Each of the behaviors or attitudes listed in the table above are associated with successful college mathematics exam preparation. This means that students who use these strategies or are open to these beliefs pass their college math courses. Compute your total score and share your score with your study group in class. Be supportive of your fellow students and offer encouragement!

Total score = _____

- Based on this survey, create a list of the top 5 test preparation and taking strategies that you currently utilize, and feel are most helpful to you.
 -
 -
 -
 -
 -
- Based on this survey, create a list of the top 5 test preparation and taking strategies that interest you, and that you feel could be most helpful to you this term. Plan on implementing these strategies.
 -
 -
 -
 -
 -

Objective 2: Create a plan for success when taking mathematics exams.

- It's important to take the opportunity to reflect on your past experiences in taking math exams as you begin a new term. We can learn a lot from these reflections and thus work toward developing a strategy for improvement. In the table below list 5 challenges you have had in past math courses when taking an exam and list a possible solution that you could try this semester.

Challenge:	Possible Solution:
1. _____	
2. _____	
3. _____	
4. _____	
5. _____	

- Develop your plan for success. Keep in mind the idea of mindsets and try to approach your test taking strategies with a growth mindset. Now is the time for growth as you begin a new term. Share your plan with your study group

2 Functions

2.1 transformations (activities from Active Prelude to Calculus section 1.8)

2.2 inverse functions

2.2.1 inverse functions (OpenStax College Algebra with Corequisite Support)

2.2.2 inverse functions activities from Active Prelude to Calculus section 1.7

Calculus Fun Fact: In Latin, **calculus** means “pebble.” Because the Romans used pebbles to do addition and subtraction on a counting board, the word became associated with computation.

**2.1 transformations (activities from Active Prelude to
Calculus section 1.8)**

1.8 Transformations of Functions

Preview Activity 1.8.1. Open a new *Desmos* graph and define the function $f(x) = x^2$. Adjust the window so that the range is for $-4 \leq x \leq 4$ and $-10 \leq y \leq 10$.

- a. In *Desmos*, define the function $g(x) = f(x) + a$. (That is, in *Desmos* on line 2, enter $g(x) = f(x) + a$.) You will get prompted to add a slider for a . Do so.

Explore by moving the slider for a and write at least one sentence to describe the effect that changing the value of a has on the graph of g .

- b. Next, define the function $h(x) = f(x - b)$. (That is, in *Desmos* on line 4, enter $h(x) = f(x - b)$ and add the slider for b .)

Move the slider for b and write at least one sentence to describe the effect that changing the value of b has on the graph of h .

- c. Now define the function $p(x) = cf(x)$. (That is, in *Desmos* on line 6, enter $p(x) = cf(x)$ and add the slider for c .)

Move the slider for c and write at least one sentence to describe the effect that changing the value of c has on the graph of p . In particular, when $c = -1$, how is the graph of p related to the graph of f ?

- d. Finally, click on the icons next to g , h , and p to temporarily hide them, and go back to Line 1 and change your formula for f . You can make it whatever you'd like, but try something like $f(x) = x^2 + 2x + 3$ or $f(x) = x^3 - 1$. Then, investigate with the sliders a , b , and c to see the effects on g , h , and p (unhiding them appropriately). Write a couple of sentences to describe your observations of your explorations.

Activity 1.8.2. Consider the functions r and s given in Figure 1.8.5 and Figure 1.8.6.

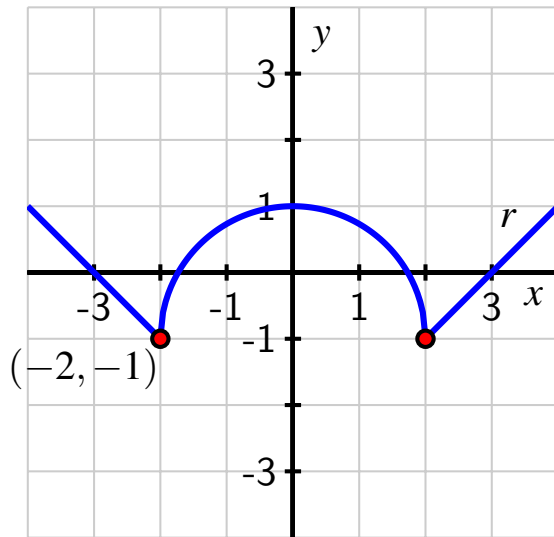


Figure 1.8.5: A parent function r .

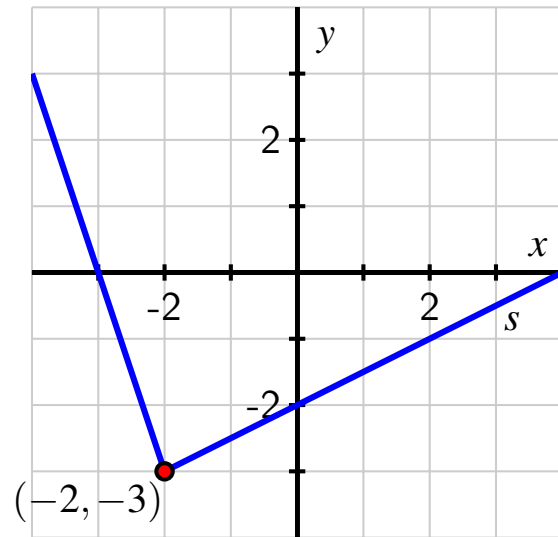


Figure 1.8.6: A parent function s .

- On the same axes as the plot of $y = r(x)$, sketch the following graphs: $y = g(x) = r(x) + 2$, $y = h(x) = r(x + 1)$, and $y = f(x) = r(x + 1) + 2$. Be sure to label the point on each of g , h , and f that corresponds to $(-2, -1)$ on the original graph of r . In addition, write one sentence to explain the overall transformations that have resulted in g , h , and f .
- Is it possible to view the function f in (a) as the result of composition of g and h ? If so, in what order should g and h be composed in order to produce f ?
- On the same axes as the plot of $y = s(x)$, sketch the following graphs: $y = k(x) = s(x) - 1$, $y = j(x) = s(x - 2)$, and $y = m(x) = s(x - 2) - 1$. Be sure to label the point on each of k , j , and m that corresponds to $(-2, -3)$ on the original graph of r . In addition, write one sentence to explain the overall transformations that have resulted in k , j , and m .
- Now consider the function $q(x) = x^2$. Determine a formula for the function that is given by $p(x) = q(x + 3) - 4$. How is p a transformation of q ?

Activity 1.8.3. Consider the functions r and s given in Figure 1.8.11 and Figure 1.8.12.

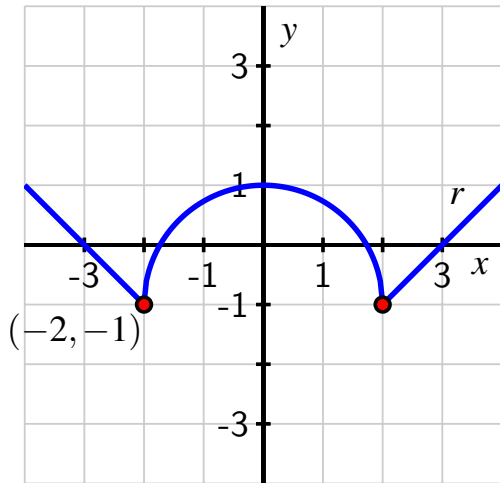


Figure 1.8.11: A parent function r .

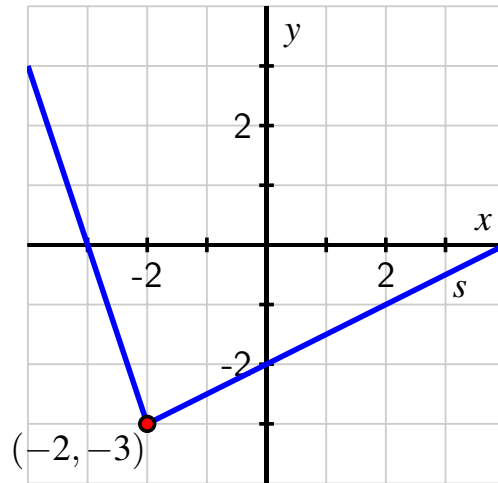
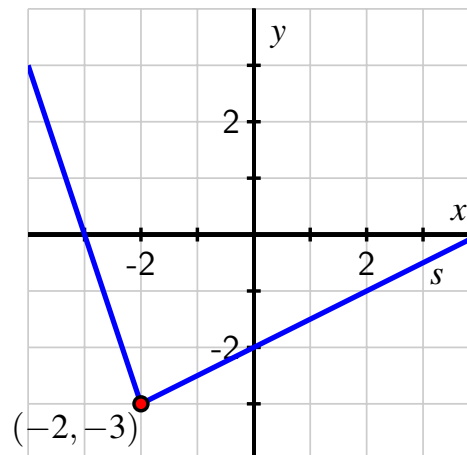
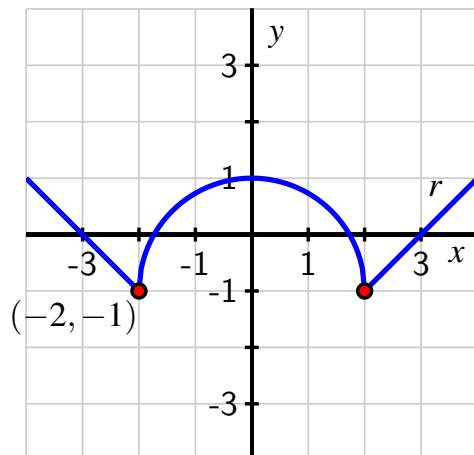


Figure 1.8.12: A parent function s .

- On the same axes as the plot of $y = r(x)$, sketch the following graphs: $y = g(x) = 3r(x)$ and $y = h(x) = \frac{1}{3}r(x)$. Be sure to label several points on each of r , g , and h with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in g and h from r .
- On the same axes as the plot of $y = s(x)$, sketch the following graphs: $y = k(x) = -s(x)$ and $y = j(x) = -\frac{1}{2}s(x)$. Be sure to label several points on each of s , k , and j with arrows to indicate their correspondence. In addition, write one sentence to explain the overall transformations that have resulted in k and j from s .
- On the additional copies of the two figures below, sketch the graphs of the following transformed functions: $y = m(x) = 2r(x + 1) - 1$ (at left) and $y = n(x) = \frac{1}{2}s(x - 2) + 2$. As above, be sure to label several points on each graph and indicate their correspondence to points on the original parent function.



- Describe in words how the function $y = m(x) = 2r(x + 1) - 1$ is the result of three elementary transformations of $y = r(x)$. Does the order in which these transformations occur matter? Why or why not?

Activity 1.8.4. Consider the functions f and g given in Figure 1.8.17 and Figure 1.8.18.

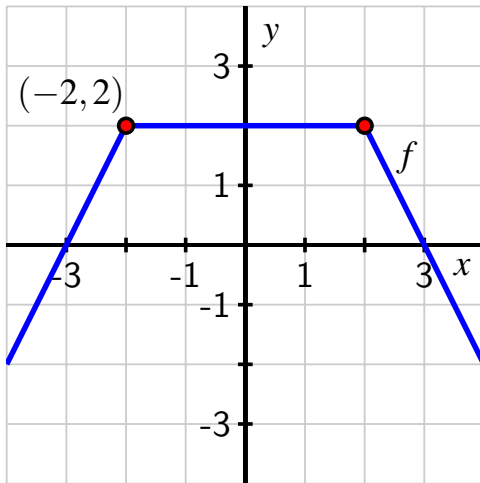


Figure 1.8.17: A parent function f .

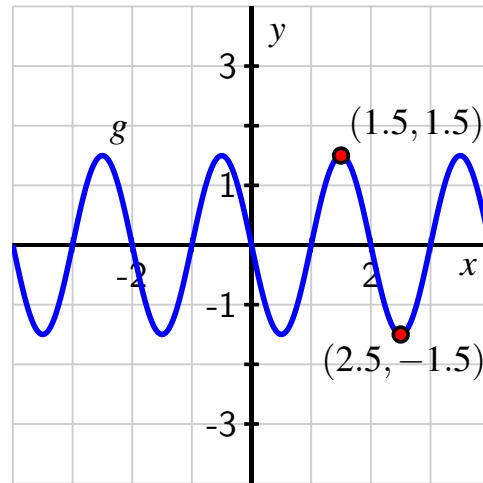
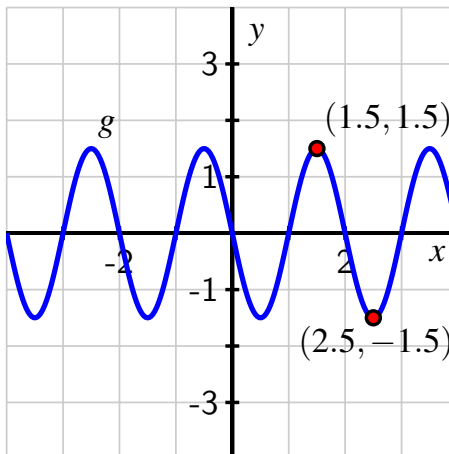


Figure 1.8.18: A parent function g .

- Sketch an accurate graph of the transformation $y = p(x) = -\frac{1}{2}f(x-1)+2$. Write at least one sentence to explain how you developed the graph of p , and identify the point on p that corresponds to the original point $(-2, 2)$ on the graph of f .
- Sketch an accurate graph of the transformation $y = q(x) = 2g(x + 0.5) - 0.75$. Write at least one sentence to explain how you developed the graph of p , and identify the point on q that corresponds to the original point $(1.5, 1.5)$ on the graph of g .
- Is the function $y = r(x) = \frac{1}{2}(-f(x-1) - 4)$ the same function as p or different? Why? Explain in two different ways: discuss the algebraic similarities and differences between p and r , and also discuss how each is a transformation of f .
- Find a formula for a function $y = s(x)$ (in terms of g) that represents this transformation of g : a horizontal shift of 1.25 units left, followed by a reflection across the x -axis and a vertical stretch of 2.5 units, followed by a vertical shift of 1.75 units. Sketch an accurate, labeled graph of s on the following axes along with the given parent function g .



2.2 inverse functions

2.2.1 inverse functions (OpenStax College Algebra with Corequisite Support)

2.2.2 inverse functions activities from Active Prelude to Calculus section 1.7

41. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.
42. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

3.7 Inverse Functions

Learning Objectives

In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

COREQUISITE SKILLS

Learning Objectives

1. Find and evaluate composite functions (IA 10.1.1).
2. Determine whether a function is one-to-one (IA 10.1.2).

Objective 1: Find and evaluate composite functions (IA 10.1.1).

A **composite** function is a two-step function and can have numerical or variable inputs.



$(f \circ g)(x) = f(g(x))$ is read as "f of g of x".

To evaluate a composite function, we always start by evaluating the inner function and then evaluate the outer function in terms of the inner function.

EXAMPLE 1

Find and evaluate composite functions.

For functions $f(x) = 2x - 7$, $g(x) = \frac{x+7}{2}$, find:

- (a) $g(5)$ (b) $f(g(5))$ (c) $f(g(x))$

✓ **Solution**

(a) To find $g(5)$, we evaluate $g(x)$ when x is 5.

$$g(x) = \frac{x+7}{2}$$

$$g(5) = \frac{5+7}{2} = \frac{12}{2} = 6$$

(b) To find $f(g(5))$, we start evaluating the inner function g in terms of 5 (see part a) and then evaluate the outer function f in terms of this value.

$$f(g(5)) = f(6) = 2(6) - 7 = 12 - 7 = 5$$

© In parts a and b we had numerical outputs because our inputs were numbers. When we find $f(g(x))$ this will be a function written in terms of the variable x .

$$f(g(x)) = f\left(\frac{x+7}{2}\right) = 2\left(\frac{x+7}{2}\right) - 7 = x + 7 - 7 = x$$

This is interesting, notice that the functions $f(x)$ and $g(x)$ have a special relationship in that one undoes the other. We call functions like this, **inverses** of one another. For any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$.

Practice Makes Perfect

Find and evaluate composite functions.

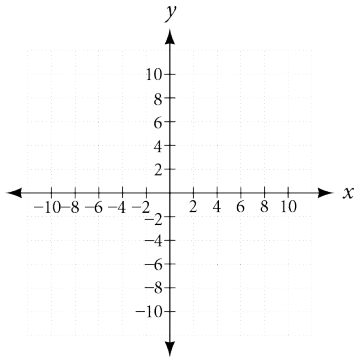
For each of the following function pairs find:

1. $f(x) = \sqrt[3]{x-2}$, $g(x) = x^3 + 2$

(a) $f(g(x))$

(b) $g(f(x))$

(c) Graph the functions $f(x)$ and $g(x)$ on the same coordinate system below



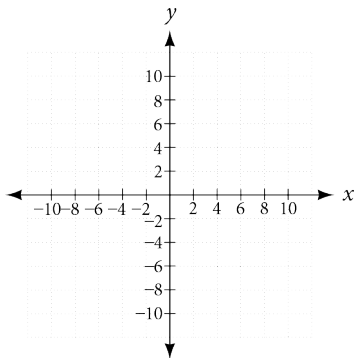
(d) What do you notice about the relationship between the graphs of $f(x)$ and $g(x)$?

2. $f(x) = \frac{1}{x+3}$, $g(x) = \frac{1}{x} - 3$

(a) $f(g(x))$

(b) $g(f(x))$

(c) Graph the functions $f(x)$ and $g(x)$ on the same coordinate system below



(d) What do you notice about the relationship between the graphs of $f(x)$ and $g(x)$?

Objective 2: Determine whether a function is one-to-one (IA 10.1.2).

In creating a process called a function, $f(x)$, it is often useful to undo this process, or create an inverse to the function, $f^{-1}(x)$. When finding the inverse, we restrict our work to one-to-one functions, this means that the inverse we find should also be one-to-one. Remember that the horizontal line test is a great way to check to see if a graph represents a one-to-one function.

For any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

The following key terms will be important to our understanding of functions and their inverses.

Function: a relation in which each input value yields a unique output value.

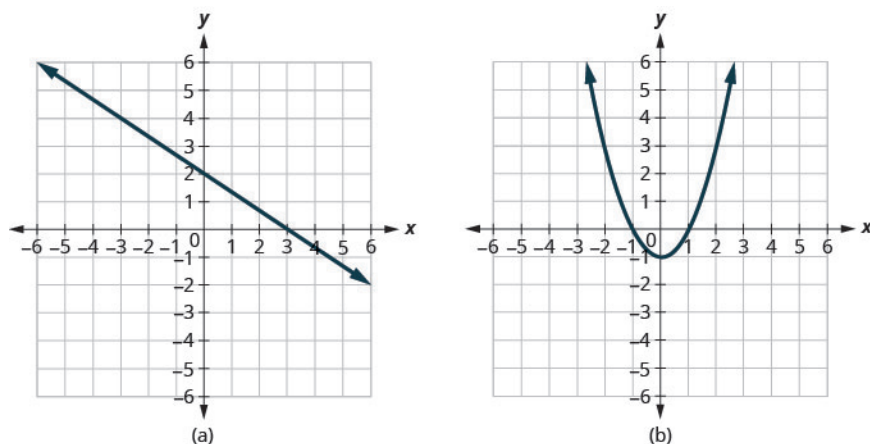
Vertical line test: a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once.

One-to-one function: a function for which each value of the output is associated with a unique input value.

Horizontal line test: a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once.

EXAMPLE 2**Determine whether a function is one-to-one.**

Determine (a) whether each graph is the graph of a function and, if so, (b) whether it is one-to-one.

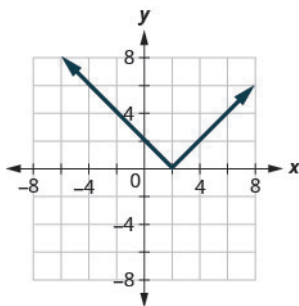
**✓ Solution**

- (a) Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.
- (b) Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. However, a horizontal line shown on the graph may intersect it in two points. This graph does not represent a one-to-one function.

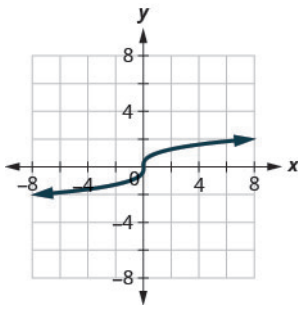
Practice Makes Perfect

Determine whether each graph is the graph of a function and, if so, whether it is one-to-one.

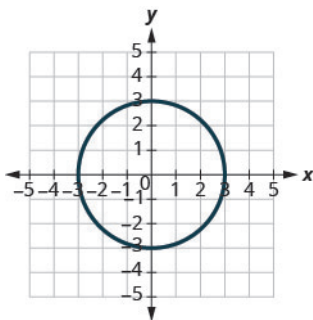
3.



4.



5.



A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function “machines” we have been studying can also run backwards. [Figure 1](#) provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

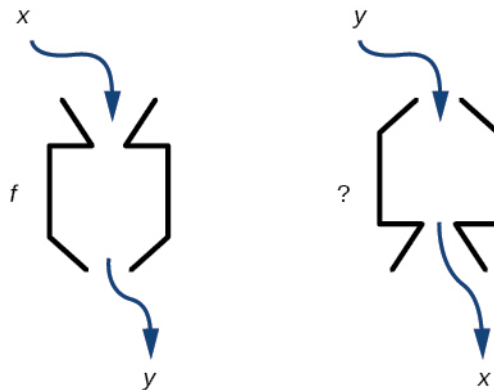


Figure 1 Can a function “machine” operate in reverse?

Verifying That Two Functions Are Inverse Functions

Betty is traveling to Milan for a fashion show and wants to know what the temperature will be. She is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, Betty wants to convert 75 degrees Fahrenheit to degrees Celsius using the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ\text{C}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, Betty gets the week’s weather forecast

from [Figure 2](#) for Milan, and wants to convert all of the temperatures to degrees Fahrenheit.

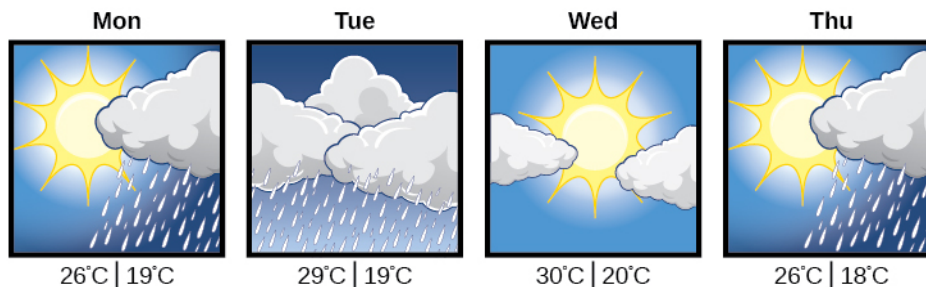


Figure 2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C . For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F - 32) \\ 26 \cdot \frac{9}{5} &= F - 32 \\ F &= 26 \cdot \frac{9}{5} + 32 \approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as “ f inverse of x .” The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The “exponent-like” notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a , so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

This holds for all x in the domain of f . Informally, this means that inverse functions “undo” each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Inverse Function

For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f . It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f .

The notation f^{-1} is read “ f inverse.” Like any other function, we can use any variable name as the input for f^{-1} , so we will often write $f^{-1}(x)$, which we read as “ f inverse of x .” Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

EXAMPLE 1

Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) &= 4, \text{ then } f^{-1}(4) = 2; \\ f(5) &= 12, \text{ then } f^{-1}(12) = 5. \end{aligned}$$


Alternatively, if we want to name the inverse function g , then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See [Table 1](#).

$(x, f(x))$	$(x, g(x))$
(2, 4)	(4, 2)
(5, 12)	(12, 5)

Table 1

 **TRY IT** #1 Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h ?



HOW TO

Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

- Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
- If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

EXAMPLE 2**Testing Inverse Relationships Algebraically**

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

✓ **Solution**

$$\begin{aligned} g(f(x)) &= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

🕒 **Analysis**

Notice the inverse operations are in reverse order of the operations from the original function.

> **TRY IT #2** If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

EXAMPLE 3**Determining Inverse Relationships for Power Functions**

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

✓ **Solution**

$$f(g(x)) = \frac{x^3}{27} \neq x$$

No, the functions are not inverses.

🕒 **Analysis**

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

> **TRY IT #3** If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1} , so the range of f is also the domain of f^{-1} . Likewise, because the inputs to f are the outputs of f^{-1} , the domain of f is the range of f^{-1} . We can visualize the situation as in [Figure 3](#).

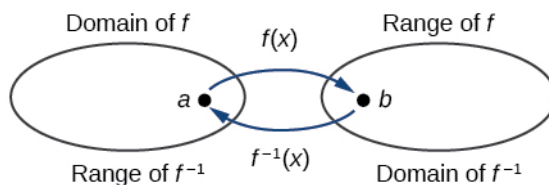


Figure 3 Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square “undoes” a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the “inverse” is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of $f = \text{range of } f^{-1} = [1, \infty)$.
- The domain of $f^{-1} = \text{range of } f = [0, \infty)$.



Q&A Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h , both meet the definition of being inverses of another function f , then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.



HOW TO

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

EXAMPLE 4

Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in [Table 2](#). We restrict the domain in such a fashion that the function assumes all y -values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube root	Square root	Absolute value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = x $	

Table 2

✓ **Solution**

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no inverse.

The absolute value function can be restricted to the domain $[0, \infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

© **Analysis**

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

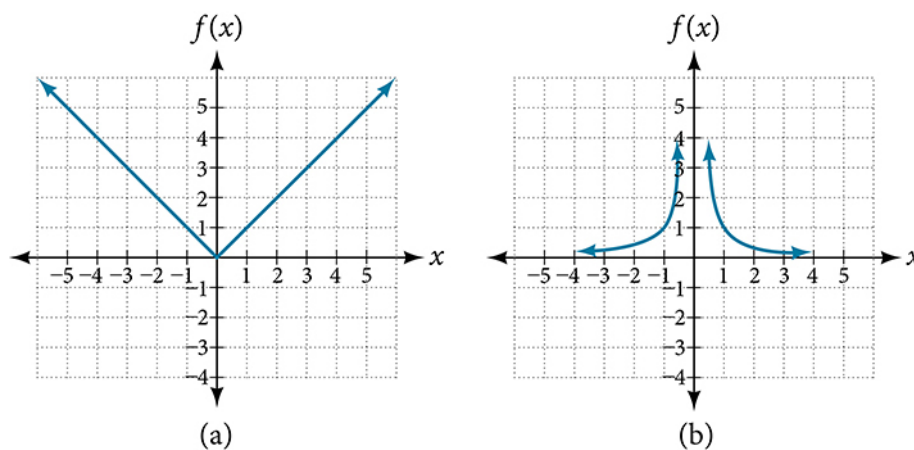


Figure 4 (a) Absolute value (b) Reciprocal square

> **TRY IT** #4 The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

EXAMPLE 5**Interpreting the Inverse of a Tabular Function**

A function $f(t)$ is given in [Table 3](#), showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Table 3**✓ Solution**

The inverse function takes an output of f and returns an input for f . So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f , 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

> **TRY IT** #5 Using [Table 4](#), find and interpret **(a)** $f(60)$, and **(b)** $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Table 4**Evaluating the Inverse of a Function, Given a Graph of the Original Function**

We saw in [Functions and Function Notation](#) that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

**HOW TO**

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y -axis of the given graph.
2. Read the inverse function's output from the x -axis of the given graph.

EXAMPLE 6**Evaluating a Function and Its Inverse from a Graph at Specific Points**

A function $g(x)$ is given in [Figure 5](#). Find $g(3)$ and $g^{-1}(3)$.

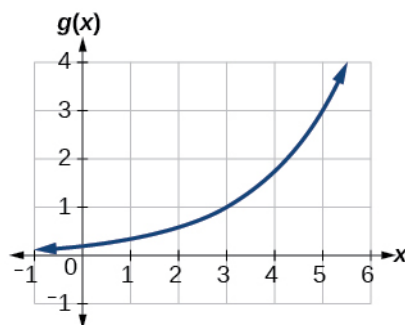


Figure 5

✓ Solution

To evaluate $g(3)$, we find 3 on the x -axis and find the corresponding output value on the y -axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See [Figure 6](#).

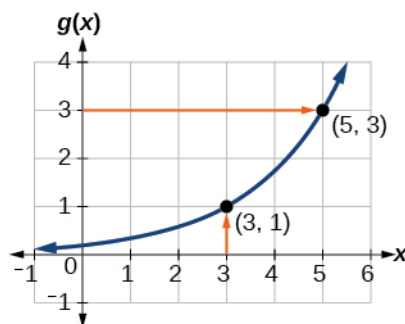


Figure 6

> TRY IT #6 Using the graph in [Figure 6](#), **(a)** find $g^{-1}(1)$, and **(b)** estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x —we can often find the inverse function by solving to obtain x as a function of y .



HOW TO

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x .
3. Interchange x and y .

EXAMPLE 7

Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32)$$

✓ **Solution**

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ C \cdot \frac{9}{5} &= F - 32 \\ F &= \frac{9}{5}C + 32 \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

> **TRY IT** #7 Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$.

EXAMPLE 8

Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

✓ **Solution**

$$\begin{array}{ll} y = \frac{2}{x-3} + 4 & \text{Set up an equation.} \\ y - 4 = \frac{2}{x-3} & \text{Subtract 4 from both sides.} \\ x - 3 = \frac{2}{y-4} & \text{Multiply both sides by } x - 3 \text{ and divide by } y - 4. \\ x = \frac{2}{y-4} + 3 & \text{Add 3 to both sides.} \end{array}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

⊙ **Analysis**

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating [Table 5](#).

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Table 5

EXAMPLE 9

Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

✓ **Solution**

$$\begin{aligned}y &= 2 + \sqrt{x - 4} \\(y - 2)^2 &= x - 4 \\x &= (y - 2)^2 + 4\end{aligned}$$

So $f^{-1}(x) = (x - 2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

⊙ **Analysis**

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x . However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f .

> **TRY IT** #8 What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in [Figure 7](#).

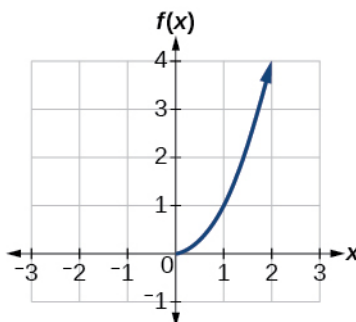


Figure 7 Quadratic function with domain restricted to $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x -axis for the input to both f and f^{-1} ?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in [Figure 8](#).

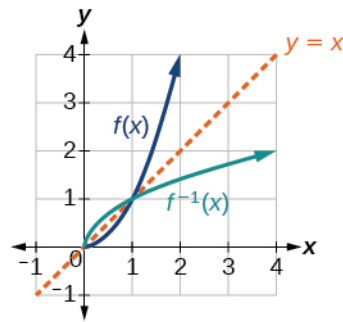


Figure 8 Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

EXAMPLE 10

Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in [Figure 9](#), sketch a graph of $f^{-1}(x)$.

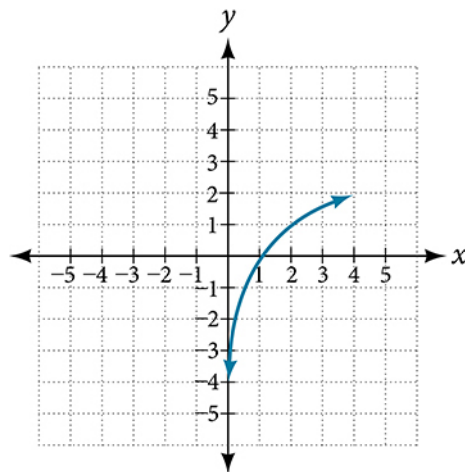


Figure 9

✓ Solution

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives [Figure 10](#).

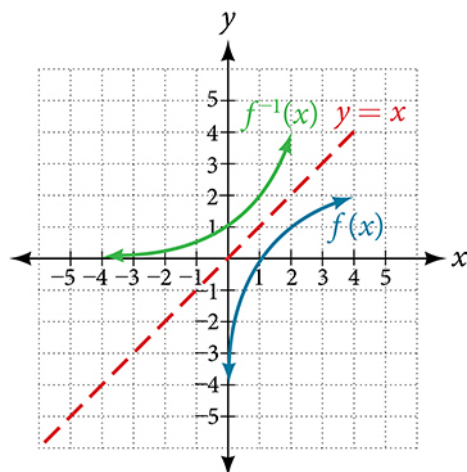


Figure 10 The function and its inverse, showing reflection about the identity line

TRY IT #9 Draw graphs of the functions f and f^{-1} from [Example 8](#).

Q&A Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

MEDIA

Access these online resources for additional instruction and practice with inverse functions.

[Inverse Functions \(http://openstax.org//inversefunction\)](http://openstax.org//inversefunction)

[One-to-one Functions \(http://openstax.org//onetooone\)](http://openstax.org//onetooone)

[Inverse Function Values Using Graph \(http://openstax.org//inversfuncgraph\)](http://openstax.org//inversfuncgraph)

[Restricting the Domain and Finding the Inverse \(http://openstax.org//restrictdomain\)](http://openstax.org//restrictdomain)



3.7 SECTION EXERCISES

Verbal

- Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?
- Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?
- Can a function be its own inverse? Explain.
- Are one-to-one functions either always increasing or always decreasing? Why or why not?
- How do you find the inverse of a function algebraically?

Algebraic

6. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a .

For the following exercises, find $f^{-1}(x)$ for each function.

7. $f(x) = x + 3$ 8. $f(x) = x + 5$ 9. $f(x) = 2 - x$
10. $f(x) = 3 - x$ 11. $f(x) = \frac{x}{x+2}$ 12. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

13. $f(x) = (x + 7)^2$ 14. $f(x) = (x - 6)^2$ 15. $f(x) = x^2 - 5$
16. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:
- Find $f(g(x))$ and $g(f(x))$.
 - What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

17. $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$ 18. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

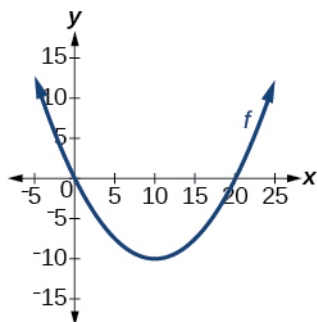
Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

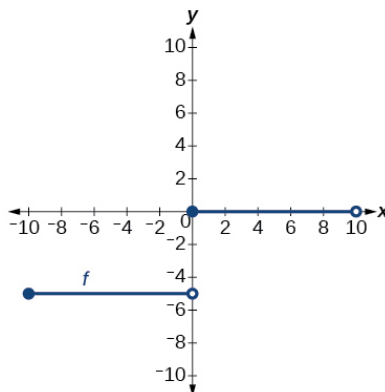
19. $f(x) = \sqrt{x}$ 20. $f(x) = \sqrt[3]{3x+1}$ 21. $f(x) = -5x + 1$
22. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

23.



24.



For the following exercises, use the graph of f shown in Figure 11.

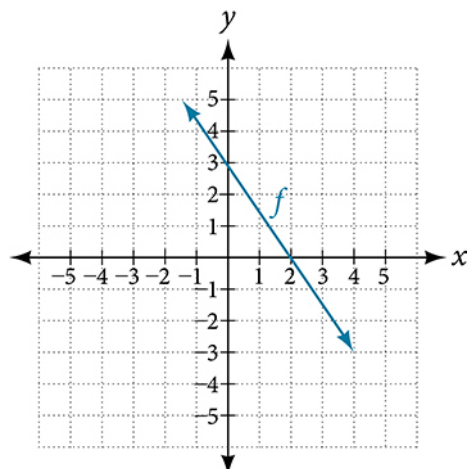


Figure 11

25. Find $f(0)$. 26. Solve $f(x) = 0$. 27. Find $f^{-1}(0)$.
28. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 12.

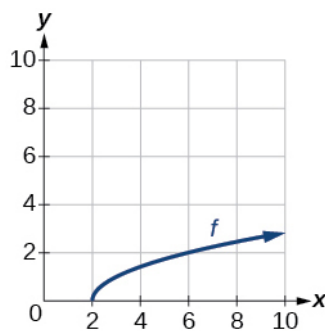


Figure 12

29. Sketch the graph of f^{-1} . 30. Find $f(6)$ and $f^{-1}(2)$. 31. If the complete graph of f is shown, find the domain of f .
32. If the complete graph of f is shown, find the range of f .

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

33. If $f(6) = 7$, find $f^{-1}(7)$. 34. If $f(3) = 2$, find $f^{-1}(2)$. 35. If $f^{-1}(-4) = -8$, find $f(-8)$.

36. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in [Table 6](#) to evaluate or solve.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

Table 6

37. Find $f(1)$.
38. Solve $f(x) = 3$.
39. Find $f^{-1}(0)$.
40. Solve $f^{-1}(x) = 7$.
41. Use the tabular representation of f in [Table 7](#) to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Table 7

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

42. $f(x) = \frac{3}{x-2}$
43. $f(x) = x^3 - 1$
44. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

45. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.
46. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.
47. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t , in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

1.7 Inverse Functions

Preview Activity 1.7.1. Recall that $F = g(C) = \frac{9}{5}C + 32$ is the function that takes Celsius temperature inputs and produces the corresponding Fahrenheit temperature outputs.

- Show that it is possible to solve the equation $F = \frac{9}{5}C + 32$ for C in terms of F and that doing so results in the equation $C = \frac{5}{9}(F - 32)$.
- Note that the equation $C = \frac{5}{9}(F - 32)$ expresses C as a function of F . Call this function h so that $C = h(F) = \frac{5}{9}(F - 32)$.
Find the simplest expression that you can for the composite function $j(C) = h(g(C))$.
- Find the simplest expression that you can for the composite function $k(F) = g(h(F))$.
- Why are the functions j and k so simple? Explain by discussing how the functions g and h process inputs to generate outputs and what happens when we first execute one followed by the other.

Activity 1.7.2. Recall Dolbear's function $F = D(N) = 40 + \frac{1}{4}N$ that converts the number, N , of snowy tree cricket chirps per minute to a corresponding Fahrenheit temperature. We have earlier established that the domain of D is $[40, 180]$ and the range of D is $[50, 85]$, as seen in Figure 1.2.3.

- Solve the equation $F = 40 + \frac{1}{4}N$ for N in terms of F . Call the resulting function $N = E(F)$.
- Explain in words the process or effect of the function $N = E(F)$. What does it take as input? What does it generate as output?
- Use the function E that you found in (a.) to compute $j(N) = E(D(N))$. Simplify your result as much as possible. Do likewise for $k(F) = D(E(F))$. What do you notice about these two composite functions j and k ?
- Consider the equations $F = 40 + \frac{1}{4}N$ and $N = 4(F - 40)$. Do these equations express different relationships between F and N , or do they express the same relationship in two different ways? Explain.

Activity 1.7.3. Determine, with justification, whether each of the following functions has an inverse function. For each function that has an inverse function, give two examples of values of the inverse function by writing statements such as “ $s^{-1}(3) = 1$ ”.

- a. The function $f : S \rightarrow S$ given by Table 1.7.11, where $S = \{0, 1, 2, 3, 4\}$.

x	0	1	2	3	4
$f(x)$	1	2	4	3	2

Table 1.7.11: Values of $y = f(x)$.

- b. The function $g : S \rightarrow S$ given by Table 1.7.12, where $S = \{0, 1, 2, 3, 4\}$.

x	0	1	2	3	4
$f(x)$	4	0	3	1	2

Table 1.7.12: Values of $y = g(x)$.

- c. The function p given by $p(t) = 7 - \frac{3}{5}t$. Assume that the domain and codomain of p are both “all real numbers”.
- d. The function q given by $q(t) = 7 - \frac{3}{5}t^4$. Assume that the domain and codomain of q are both “all real numbers”.
- e. The functions r and s given by the graphs in Figure 1.7.13 and Figure 1.7.14. Assume that the graphs show all of the important behavior of the functions and that the apparent trends continue beyond what is pictured.

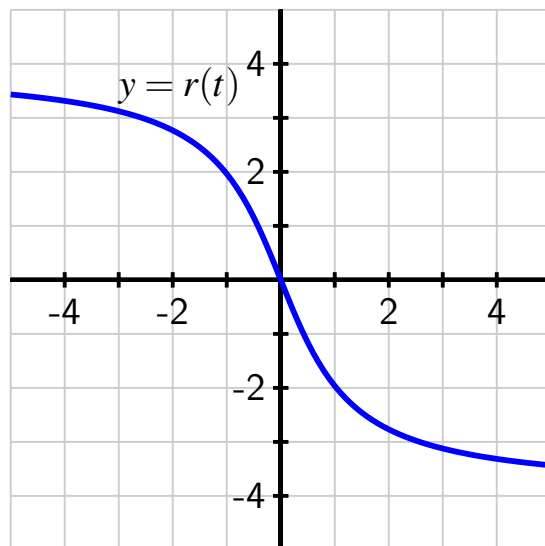


Figure 1.7.13: The graph of $y = r(t)$.

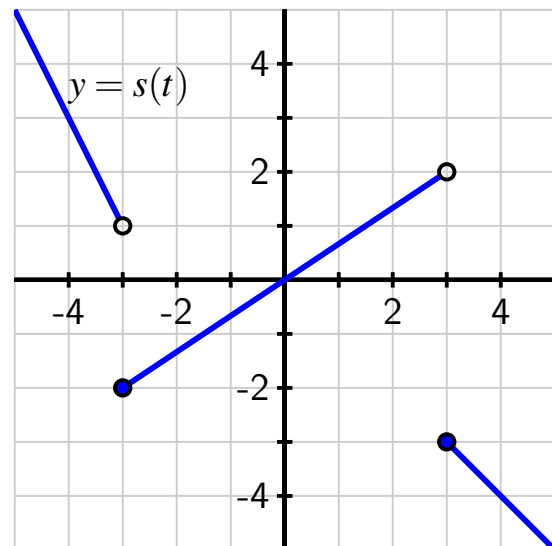


Figure 1.7.14: The graph of $y = s(t)$.

Activity 1.7.4. During a major rainstorm, the rainfall at Gerald R. Ford Airport is measured on a frequent basis for a 10-hour period of time. The following function g models the *rate*, R , at which the rain falls (in cm/hr) on the time interval $t = 0$ to $t = 10$:

$$R = g(t) = \frac{4}{t+2} + 1$$

- Compute $g(3)$ and write a complete sentence to explain its meaning in the given context, including units.
- Compute the average rate of change of g on the time interval $[3, 5]$ and write two careful complete sentences to explain the meaning of this value in the context of the problem, including units. Explicitly address what the value you compute tells you about how rain is falling over a certain time interval, and what you should expect as time goes on.
- Plot the function $y = g(t)$ using a computational device. On the domain $[1, 10]$, what is the corresponding range of g ? Why does the function g have an inverse function?
- Determine $g^{-1}\left(\frac{9}{5}\right)$ and write a complete sentence to explain its meaning in the given context.
- According to the model g , is there ever a time during the storm that the rain falls at a rate of exactly 1 centimeter per hour? Why or why not? Provide an algebraic justification for your answer.

3 Topics for Limits

- 3.1 real numbers and algebra foundations (OpenStax College Algebra with Corequisite Support)
 - 3.1.1 classification of real numbers
 - 3.1.2 order of operations
 - 3.1.3 properties of real numbers
 - 3.1.4 evaluating and simplifying algebraic expressions
- 3.2 working with roots
 - 3.2.1 simplifying radical expressions (OpenStax College Algebra with Corequisite Support)
 - 3.2.2 rationalizing denominators/working with radical expressions (Modeling, Functions, and Graphs)
- 3.3 polynomials
 - 3.3.1 adding/subtracting/multiplying polynomials (OpenStax College Algebra with Corequisite Support)
 - 3.3.2 factoring polynomials (OpenStax College Algebra with Corequisite Support)
- 3.4 rational expressions (OpenStax College Algebra with Corequisite Support)
- 3.5 big and small numbers (worksheet - Benjamin Kennedy)

Calculus Fun Fact: Anyone who works with computer graphics, such as a video game programmer, uses calculus while working with vectors in which reactions and outcomes are predicted.

**3.1 real numbers and algebra foundations (OpenStax
College Algebra with Corequisite Support)**

3.1.1 classification of real numbers

3.1.2 order of operations

3.1.3 properties of real numbers

3.1.4 evaluating and simplifying algebraic expressions

understand you and your current feelings about the discipline.

3. Share your autobiographies with your study group members. This helps to create a community in the classroom when common themes emerge.

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattle herders, and traders used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century CE in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century CE, negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, \dots\}$ where the ellipsis (...) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers:

$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\begin{array}{ccc} \text{negative integers} & \text{zero} & \text{positive integers} \\ \dots, -3, -2, -1, & 0, & 1, 2, 3, \dots \end{array}$$

The set of **rational numbers** is written as $\{\frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

- (a) a terminating decimal: $\frac{15}{8} = 1.875$, or (b) a repeating decimal: $\frac{4}{11} = 0.36363636 \dots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

EXAMPLE 1

Writing Integers as Rational Numbers

Write each of the following as a rational number.

- (a) 7 (b) 0 (c) -8

Solution

Write a fraction with the integer in the numerator and 1 in the denominator.

- (a) $7 = \frac{7}{1}$ (b) $0 = \frac{0}{1}$ (c) $-8 = -\frac{8}{1}$

> **TRY IT** #1 Write each of the following as a rational number.

- (a) 11 (b) 3 (c) -4

EXAMPLE 2

Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

- (a) $-\frac{5}{7}$ (b) $\frac{15}{5}$ (c) $\frac{13}{25}$

✓ **Solution**

Write each fraction as a decimal by dividing the numerator by the denominator.

- (a) $-\frac{5}{7} = -0.\overline{714285}$, a repeating decimal (b) $\frac{15}{5} = 3$ (or 3.0), a terminating decimal
 (c) $\frac{13}{25} = 0.52$, a terminating decimal

> **TRY IT** #2 Write each of the following rational numbers as either a terminating or repeating decimal.

- (a) $\frac{68}{17}$ (b) $\frac{8}{13}$ (c) $-\frac{17}{20}$

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h|h \text{ is not a rational number}\}$$

EXAMPLE 3

Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

- (a) $\sqrt{25}$ (b) $\frac{33}{9}$ (c) $\sqrt{11}$ (d) $\frac{17}{34}$ (e) 0.3033033303333 ...

✓ **Solution**

(a) $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.

(b) $\frac{33}{9}$: Because it is a fraction of integers, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}} = \frac{11}{3} = 3.\overline{6}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.

(c) $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.

(d) $\frac{17}{34}$: Because it is a fraction of integers, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34} = \frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}} = \frac{1}{2} = 0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.

e) $0.3033033303333 \dots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

> **TRY IT** #3 Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

- a) $\frac{7}{77}$ b) $\sqrt{81}$ c) $4.27027002700027 \dots$ d) $\frac{91}{13}$ e) $\sqrt{39}$

Real Numbers

Given any number n , we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or -). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line. The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in [Figure 1](#).

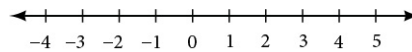


Figure 1 The real number line

EXAMPLE 4

Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

- a) $-\frac{10}{3}$ b) $\sqrt{5}$ c) $-\sqrt{289}$ d) -6π e) $0.615384615384 \dots$

✓ Solution

- a) $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
 b) $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
 c) $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
 d) -6π is negative and irrational. It lies to the left of 0.
 e) $0.615384615384 \dots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

> **TRY IT** #4 Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

- a) $\sqrt{73}$ b) $-11.411411411 \dots$ c) $\frac{47}{19}$ d) $-\frac{\sqrt{5}}{2}$ e) 6.210735

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as [Figure 2](#).

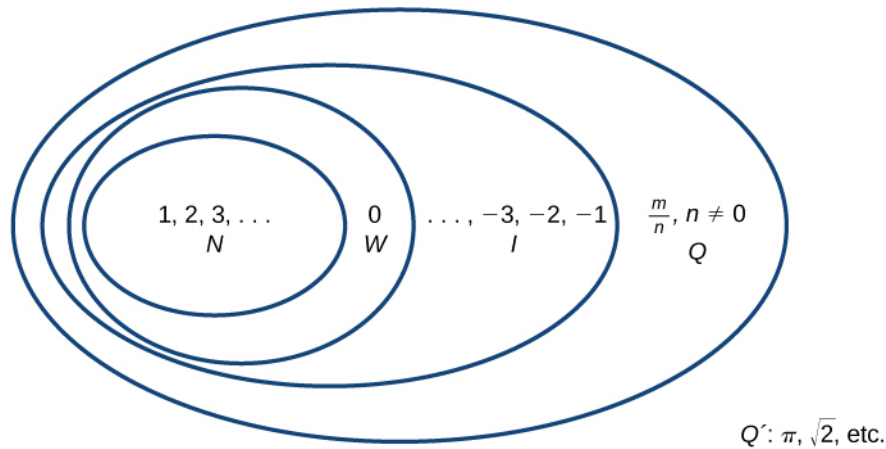


Figure 2 Sets of numbers
 N : the set of natural numbers
 W : the set of whole numbers
 I : the set of integers
 Q : the set of rational numbers
 Q' : the set of irrational numbers

Sets of Numbers

The set of **natural numbers** includes the numbers used for counting: $\{1, 2, 3, \dots\}$.

The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \dots\}$.

The set of **integers** adds the negative natural numbers to the set of whole numbers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

The set of **rational numbers** includes fractions written as $\{\frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0\}$.

The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \mid h \text{ is not a rational number}\}$.

EXAMPLE 5

Differentiating the Sets of Numbers

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

- Ⓐ $\sqrt{36}$ Ⓑ $\frac{8}{3}$ Ⓒ $\sqrt{73}$ Ⓓ -6 Ⓔ $3.2121121112\dots$

✓ **Solution**

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\bar{6}$				X	
c. $\sqrt{73}$					X
d. -6			X	X	
e. $3.2121121112\dots$					X

> **TRY IT** #5 Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

- (a) $-\frac{35}{7}$ (b) 0 (c) $\sqrt{169}$ (d) $\sqrt{24}$ (e) 4.763763763 ...

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.

$$a^n = \overset{n \text{ factors}}{a \cdot a \cdot a \cdot \dots \cdot a}$$

In this notation, a^n is read as the n th power of a , or a to the n where a is called the **base** and n is called the **exponent**. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \cdot \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses (), brackets [], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$\begin{aligned} 24 + 6 \cdot \frac{2}{3} - 4^2 \\ 24 + 6 \cdot \frac{2}{3} - 16 \end{aligned}$$

Next, perform multiplication or division, left to right.

$$\begin{aligned} 24 + 6 \cdot \frac{2}{3} - 16 \\ 24 + 4 - 16 \end{aligned}$$

Lastly, perform addition or subtraction, left to right.

$$\begin{aligned} 24 + 4 - 16 \\ 28 - 16 \\ 12 \end{aligned}$$

Therefore, $24 + 6 \cdot \frac{2}{3} - 4^2 = 12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

Order of Operations

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the

acronym **PEMDAS**:

P(arentheses)

E(xponents)

M(ultiplication) and **D**(ivision)

A(ddition) and **S**(ubtraction)



HOW TO

Given a mathematical expression, simplify it using the order of operations.

Step 1. Simplify any expressions within grouping symbols.

Step 2. Simplify any expressions containing exponents or radicals.

Step 3. Perform any multiplication and division in order, from left to right.

Step 4. Perform any addition and subtraction in order, from left to right.

EXAMPLE 6

Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

(a) $(3 \cdot 2)^2 - 4(6 + 2)$ (b) $\frac{5^2-4}{7} - \sqrt{11-2}$ (c) $6 - |5 - 8| + 3(4 - 1)$ (d) $\frac{14-3 \cdot 2}{2 \cdot 5-3^2}$

(e) $7(5 \cdot 3) - 2[(6 - 3) - 4^2] + 1$

✓ **Solution**

(a)

$$\begin{aligned} (3 \cdot 2)^2 - 4(6 + 2) &= (6)^2 - 4(8) && \text{Simplify parentheses.} \\ &= 36 - 4(8) && \text{Simplify exponent.} \\ &= 36 - 32 && \text{Simplify multiplication.} \\ &= 4 && \text{Simplify subtraction.} \end{aligned}$$

(b)

$$\begin{aligned} \frac{5^2-4}{7} - \sqrt{11-2} &= \frac{5^2-4}{7} - \sqrt{9} && \text{Simplify grouping symbols (radical).} \\ &= \frac{5^2-4}{7} - 3 && \text{Simplify radical.} \\ &= \frac{25-4}{7} - 3 && \text{Simplify exponent.} \\ &= \frac{21}{7} - 3 && \text{Simplify subtraction in numerator.} \\ &= 3 - 3 && \text{Simplify division.} \\ &= 0 && \text{Simplify subtraction.} \end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

(c)

$$\begin{aligned} 6 - |5 - 8| + 3(4 - 1) &= 6 - |-3| + 3(3) && \text{Simplify inside grouping symbols.} \\ &= 6 - 3 + 3(3) && \text{Simplify absolute value.} \\ &= 6 - 3 + 9 && \text{Simplify multiplication.} \\ &= 3 + 9 && \text{Simplify subtraction.} \\ &= 12 && \text{Simplify addition.} \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \quad \frac{14-3 \cdot 2}{2 \cdot 5-3^2} &= \frac{14-3 \cdot 2}{2 \cdot 5-9} && \text{Simplify exponent.} \\
 &= \frac{14-6}{10-9} && \text{Simplify products.} \\
 &= \frac{8}{1} && \text{Simplify differences.} \\
 &= 8 && \text{Simplify quotient.}
 \end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

$$\begin{aligned}
 \textcircled{e} \quad 7(5 \cdot 3) - 2[(6-3) - 4^2] + 1 &= 7(15) - 2[(3) - 4^2] + 1 && \text{Simplify inside parentheses.} \\
 &= 7(15) - 2(3 - 16) + 1 && \text{Simplify exponent.} \\
 &= 7(15) - 2(-13) + 1 && \text{Subtract.} \\
 &= 105 + 26 + 1 && \text{Multiply.} \\
 &= 132 && \text{Add.}
 \end{aligned}$$

TRY IT #6 Use the order of operations to evaluate each of the following expressions.

$$\begin{aligned}
 \textcircled{a} \quad \sqrt{5^2 - 4^2} + 7(5 - 4)^2 & \quad \textcircled{b} \quad 1 + \frac{7 \cdot 5 - 8 \cdot 4}{9 - 6} & \quad \textcircled{c} \quad |1.8 - 4.3| + 0.4\sqrt{15 + 10} \\
 \textcircled{d} \quad \frac{1}{2}[5 \cdot 3^2 - 7^2] + \frac{1}{3} \cdot 9^2 & \quad \textcircled{e} \quad [(3 - 8)^2 - 4] - (3 - 8)
 \end{aligned}$$

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.

$$a + b = b + a$$

We can better see this relationship when using real numbers.

$$(-2) + 7 = 5 \quad \text{and} \quad 7 + (-2) = 5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \cdot b = b \cdot a$$

Again, consider an example with real numbers.

$$(-11) \cdot (-4) = 44 \quad \text{and} \quad (-4) \cdot (-11) = 44$$

It is important to note that neither subtraction nor division is commutative. For example, $17 - 5$ is not the same as $5 - 17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc) = (ab)c$$

Consider this example.

$$(3 \cdot 4) \cdot 5 = 60 \quad \text{and} \quad 3 \cdot (4 \cdot 5) = 60$$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15 + (-9)] + 23 = 29 \quad \text{and} \quad 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

$$\begin{array}{l} 8 - (3 - 15) \stackrel{?}{=} (8 - 3) - 15 \\ 8 - (-12) = 5 - 15 \\ 20 \neq -10 \end{array} \qquad \begin{array}{l} 64 \div (8 \div 4) \stackrel{?}{=} (64 \div 8) \div 4 \\ 64 \div 2 \stackrel{?}{=} 8 \div 4 \\ 32 \neq 2 \end{array}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{array}{l} 4 \cdot [12 + (-7)] = 4 \cdot 12 + 4 \cdot (-7) \\ = 48 + (-28) \\ = 20 \end{array}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by -7 , and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{array}{l} 6 + (3 \cdot 5) \stackrel{?}{=} (6 + 3) \cdot (6 + 5) \\ 6 + (15) \stackrel{?}{=} (9) \cdot (11) \\ 21 \neq 99 \end{array}$$

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b)$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$, we add the opposite.

$$12 + (-1) \cdot (5 + 3)$$

Now, distribute -1 and simplify the result.

$$\begin{array}{l} 12 - (5 + 3) = 12 + (-1) \cdot (5 + 3) \\ = 12 + [(-1) \cdot 5 + (-1) \cdot 3] \\ = 12 + (-8) \\ = 4 \end{array}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{array}{l} 12 - (5 + 3) = 12 + (-5 - 3) \\ = 12 + (-8) \\ = 4 \end{array}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \cdot 1 = a$$

For example, we have $(-6) + 0 = -6$ and $23 \cdot 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The **inverse property of addition** states that, for every real number a , there is a unique number, called the additive inverse (or opposite), denoted by $(-a)$, that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0$$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a , there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \cdot \frac{1}{a} = 1$$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a \cdot \frac{1}{a} = \left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{2}\right) = 1$$

Properties of Real Numbers

The following properties hold for real numbers a , b , and c .

	Addition	Multiplication
Commutative Property	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Property	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Distributive Property	$a \cdot (b + c) = a \cdot b + a \cdot c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a + 0 = a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number a $a \cdot 1 = a$
Inverse Property	Every real number a has an additive inverse, or opposite, denoted $-a$, such that $a + (-a) = 0$	Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a \cdot \left(\frac{1}{a}\right) = 1$

EXAMPLE 7**Using Properties of Real Numbers**

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

- (a) $3 \cdot 6 + 3 \cdot 4$ (b) $(5 + 8) + (-8)$ (c) $6 - (15 + 9)$ (d) $\frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right)$ (e) $100 \cdot [0.75 + (-2.38)]$

✓ **Solution**

(a)

$$\begin{aligned} 3 \cdot 6 + 3 \cdot 4 &= 3 \cdot (6 + 4) && \text{Distributive property.} \\ &= 3 \cdot 10 && \text{Simplify.} \\ &= 30 && \text{Simplify.} \end{aligned}$$

(b)

$$\begin{aligned} (5 + 8) + (-8) &= 5 + [8 + (-8)] && \text{Associative property of addition.} \\ &= 5 + 0 && \text{Inverse property of addition.} \\ &= 5 && \text{Identity property of addition.} \end{aligned}$$

(c)

$$\begin{aligned} 6 - (15 + 9) &= 6 + [(-15) + (-9)] && \text{Distributive property.} \\ &= 6 + (-24) && \text{Simplify.} \\ &= -18 && \text{Simplify.} \end{aligned}$$

(d)

$$\begin{aligned} \frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right) &= \frac{4}{7} \cdot \left(\frac{7}{4} \cdot \frac{2}{3}\right) && \text{Commutative property of multiplication.} \\ &= \left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} && \text{Associative property of multiplication.} \\ &= 1 \cdot \frac{2}{3} && \text{Inverse property of multiplication.} \\ &= \frac{2}{3} && \text{Identity property of multiplication.} \end{aligned}$$

(e)

$$\begin{aligned} 100 \cdot [0.75 + (-2.38)] &= 100 \cdot 0.75 + 100 \cdot (-2.38) && \text{Distributive property.} \\ &= 75 + (-238) && \text{Simplify.} \\ &= -163 && \text{Simplify.} \end{aligned}$$

> **TRY IT** #7 Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

$$\begin{array}{lll} \text{(a)} \quad \left(-\frac{23}{5}\right) \cdot \left[11 \cdot \left(-\frac{5}{23}\right)\right] & \text{(b)} \quad 5 \cdot (6.2 + 0.4) & \text{(c)} \quad 18 - (7 - 15) \\ \text{(d)} \quad \frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right] & \text{(e)} \quad 6 \cdot (-3) + 6 \cdot 3 & \end{array}$$

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x + 5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x + 5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$\begin{aligned} (-3)^5 &= (-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) & x^5 &= x \cdot x \cdot x \cdot x \cdot x \\ (2 \cdot 7)^3 &= (2 \cdot 7) \cdot (2 \cdot 7) \cdot (2 \cdot 7) & (yz)^3 &= (yz) \cdot (yz) \cdot (yz) \end{aligned}$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

EXAMPLE 8**Describing Algebraic Expressions**

List the constants and variables for each algebraic expression.

(a) $x + 5$ (b) $\frac{4}{3}\pi r^3$ (c) $\sqrt{2m^3n^2}$

✓ **Solution**

	Constants	Variables
a. $x + 5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m, n

> **TRY IT** #8 List the constants and variables for each algebraic expression.

(a) $2\pi r(r + h)$ (b) $2(L + W)$ (c) $4y^3 + y$

EXAMPLE 9**Evaluating an Algebraic Expression at Different Values**Evaluate the expression $2x - 7$ for each value for x .

(a) $x = 0$ (b) $x = 1$ (c) $x = \frac{1}{2}$ (d) $x = -4$

✓ **Solution**

(a) Substitute 0 for x . (b) Substitute 1 for x . (c) Substitute $\frac{1}{2}$ for x . (d) Substitute -4 for x .

$$\begin{array}{rcl}
 2x - 7 & = & 2(0) - 7 \\
 & = & 0 - 7 \\
 & = & -7
 \end{array}
 \quad
 \begin{array}{rcl}
 2x - 7 & = & 2(1) - 7 \\
 & = & 2 - 7 \\
 & = & -5
 \end{array}
 \quad
 \begin{array}{rcl}
 2x - 7 & = & 2\left(\frac{1}{2}\right) - 7 \\
 & = & 1 - 7 \\
 & = & -6
 \end{array}
 \quad
 \begin{array}{rcl}
 2x - 7 & = & 2(-4) - 7 \\
 & = & -8 - 7 \\
 & = & -15
 \end{array}$$

> **TRY IT** #9 Evaluate the expression $11 - 3y$ for each value for y .

(a) $y = 2$ (b) $y = 0$ (c) $y = \frac{2}{3}$ (d) $y = -5$

EXAMPLE 10**Evaluating Algebraic Expressions**

Evaluate each expression for the given values.

(a) $x + 5$ for $x = -5$ (b) $\frac{t}{2t-1}$ for $t = 10$ (c) $\frac{4}{3}\pi r^3$ for $r = 5$ (d) $a + ab + b$ for $a = 11, b = -8$
 (e) $\sqrt{2m^3n^2}$ for $m = 2, n = 3$

✓ **Solution**

(a) Substitute -5 for x . (b) Substitute 10 for t . (c) Substitute 5 for r .

$$\begin{array}{rcl}
 x + 5 & = & (-5) + 5 \\
 & = & 0
 \end{array}
 \quad
 \begin{array}{rcl}
 \frac{t}{2t-1} & = & \frac{(10)}{2(10)-1} \\
 & = & \frac{10}{20-1} \\
 & = & \frac{10}{19}
 \end{array}
 \quad
 \begin{array}{rcl}
 \frac{4}{3}\pi r^3 & = & \frac{4}{3}\pi(5)^3 \\
 & = & \frac{4}{3}\pi(125) \\
 & = & \frac{500}{3}\pi
 \end{array}$$

Ⓓ Substitute 11 for a and -8 for b .

$$\begin{aligned} a + ab + b &= (11) + (11)(-8) + (-8) \\ &= 11 - 88 - 8 \\ &= -85 \end{aligned}$$

Ⓔ Substitute 2 for m and 3 for n .

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

> **TRY IT** #10 Evaluate each expression for the given values.

Ⓐ $\frac{y+3}{y-3}$ for $y = 5$ Ⓑ $7 - 2t$ for $t = -2$ Ⓒ $\frac{1}{3}\pi r^2$ for $r = 11$

Ⓓ $(p^2q)^3$ for $p = -2, q = 3$ Ⓔ $4(m - n) - 5(n - m)$ for $m = \frac{2}{3}, n = \frac{1}{3}$

Formulas

An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x + 1 = 7$ has the solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2(3) + 1 = 7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r , the area A can be found by evaluating the expression πr^2 .

EXAMPLE 11

Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r(r + h)$. See [Figure 3](#). Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π .

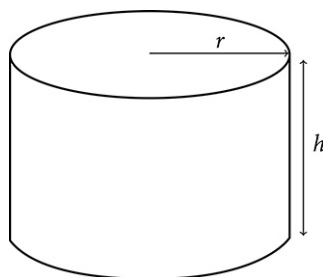


Figure 3 Right circular cylinder

✓ Solution

Evaluate the expression $2\pi r(r + h)$ for $r = 6$ and $h = 9$.

$$\begin{aligned} S &= 2\pi r(r + h) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

> **TRY IT** #11 A photograph with length L and width W is placed in a mat of width 8 centimeters (cm). The area of the mat (in square centimeters, or cm^2) is found to be $A = (L + 16)(W + 16) - L \cdot W$. See [Figure 4](#). Find the area of a mat for a photograph with length 32 cm and width 24 cm.

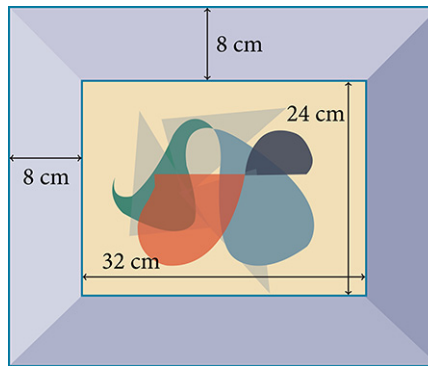


Figure 4

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

EXAMPLE 12

Simplifying Algebraic Expressions

Simplify each algebraic expression.

(a) $3x - 2y + x - 3y - 7$ (b) $2r - 5(3 - r) + 4$ (c) $(4t - \frac{5}{4}s) - (\frac{2}{3}t + 2s)$ (d) $2mn - 5m + 3mn + n$

✓ Solution

(a)

$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 \\ &= 4x - 5y - 7 \end{aligned}$$

Commutative property of addition.
Simplify.

(b)

$$\begin{aligned} 2r - 5(3 - r) + 4 &= 2r - 15 + 5r + 4 \\ &= 2r + 5r - 15 + 4 \\ &= 7r - 11 \end{aligned}$$

Distributive property.
Commutative property of addition.
Simplify.

(c)

$$\begin{aligned} (4t - \frac{5}{4}s) - (\frac{2}{3}t + 2s) &= 4t - \frac{5}{4}s - \frac{2}{3}t - 2s \\ &= 4t - \frac{2}{3}t - \frac{5}{4}s - 2s \\ &= \frac{10}{3}t - \frac{13}{4}s \end{aligned}$$

Distributive property.
Commutative property of addition.
Simplify.

(d)

$$\begin{aligned} 2mn - 5m + 3mn + n &= 2mn + 3mn - 5m + n \\ &= 5mn - 5m + n \end{aligned}$$

Commutative property of addition.
Simplify.

> **TRY IT** #12 Simplify each algebraic expression.

(a) $\frac{2}{3}y - 2(\frac{4}{3}y + z)$ (b) $\frac{5}{t} - 2 - \frac{3}{t} + 1$ (c) $4p(q - 1) + q(1 - p)$
 (d) $9r - (s + 2r) + (6 - s)$

EXAMPLE 13

Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P = L + W + L + W$. Simplify this expression.

 **Solution**

$$P = L + W + L + W$$

$$P = L + L + W + W$$


$$P = 2L + 2W$$

$$P = 2(L + W)$$

Commutative property of addition

Simplify

Distributive property

 **TRY IT** #13 If the amount P is deposited into an account paying simple interest r for time t , the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

 **MEDIA**

Access these online resources for additional instruction and practice with real numbers.

[Simplify an Expression.](http://openstax.org//simexpress) (<http://openstax.org//simexpress>)

[Evaluate an Expression 1.](http://openstax.org//ordofoper1) (<http://openstax.org//ordofoper1>)

[Evaluate an Expression 2.](http://openstax.org//ordofoper2) (<http://openstax.org//ordofoper2>)



1.1 SECTION EXERCISES

Verbal

- Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.
- What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?
- What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

Numeric

For the following exercises, simplify the given expression.

- $10 + 2 \times (5 - 3)$
- $6 \div 2 - (81 \div 3^2)$
- $18 + (6 - 8)^3$
- $-2 \times [16 \div (8 - 4)^2]^2$
- $4 - 6 + 2 \times 7$
- $3(5 - 8)$
- $4 + 6 - 10 \div 2$
- $12 \div (36 \div 9) + 6$
- $(4 + 5)^2 \div 3$
- $3 - 12 \times 2 + 19$
- $2 + 8 \times 7 \div 4$
- $5 + (6 + 4) - 11$
- $9 - 18 \div 3^2$
- $14 \times 3 \div 7 - 6$
- $9 - (3 + 11) \times 2$
- $6 + 2 \times 2 - 1$
- $64 \div (8 + 4 \times 2)$
- $9 + 4(2^2)$
- $(12 \div 3 \times 3)^2$
- $25 \div 5^2 - 7$
- $(15 - 7) \times (3 - 7)$
- $2 \times 4 - 9(-1)$
- $4^2 - 25 \times \frac{1}{5}$
- $12(3 - 1) \div 6$

Algebraic

For the following exercises, evaluate the expressions using the given variable.

28. $8(x + 3) - 64$ for $x = 2$ 29. $4y + 8 - 2y$ for $y = 3$ 30. $(11a + 3) - 18a + 4$ for $a = -2$
31. $4z - 2z(1 + 4) - 36$ for $z = 5$ 32. $4y(7 - 2)^2 + 200$ for $y = -2$ 33. $-(2x)^2 + 1 + 3$ for $x = 2$
34. For the $8(2 + 4) - 15b + b$ for $b = -3$ 35. $2(11c - 4) - 36$ for $c = 0$ 36. $4(3 - 1)x - 4$ for $x = 10$
37. $\frac{1}{4}(8w - 4^2)$ for $w = 1$

For the following exercises, simplify the expression.

38. $4x + x(13 - 7)$ 39. $2y - (4)^2y - 11$ 40. $\frac{a}{2^3}(64) - 12a \div 6$
41. $8b - 4b(3) + 1$ 42. $5l \div 3l \times (9 - 6)$ 43. $7z - 3 + z \times 6^2$
44. $4 \times 3 + 18x \div 9 - 12$ 45. $9(y + 8) - 27$ 46. $(\frac{9}{6}t - 4)2$
47. $6 + 12b - 3 \times 6b$ 48. $18y - 2(1 + 7y)$ 49. $(\frac{4}{9})^2 \times 27x$
50. $8(3 - m) + 1(-8)$ 51. $9x + 4x(2 + 3) - 4(2x + 3x)$ 52. $5^2 - 4(3x)$

Real-World Applications

For the following exercises, consider this scenario: Fred earns \$40 at the community garden. He spends \$10 on a streaming subscription, puts half of what is left in a savings account, and gets another \$5 for walking his neighbor's dog.

53. Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.
54. How much money does Fred keep?

For the following exercises, solve the given problem.

55. According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π . Is the circumference of a quarter a whole number, a rational number, or an irrational number?
56. Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

57. Write the equation that describes the situation. 58. Solve for g .

For the following exercise, solve the given problem.

59. Ramon runs the marketing department at their company. Their department gets a budget every year, and every year, they must spend the entire budget without going over. If they spend less than the budget, then the department gets a smaller budget the following year. At the beginning of this year, Ramon got \$2.5 million for the annual marketing budget. They must spend the budget such that $2,500,000 - x = 0$. What property of addition tells us what the value of x must be?

Technology

For the following exercises, use a graphing calculator to solve for x . Round the answers to the nearest hundredth.

60. $0.5(12.3)^2 - 48x = \frac{3}{5}$ 61. $(0.25 - 0.75)^2x - 7.2 = 9.9$

Extensions

62. If a whole number is not a natural number, what must the number be?
63. Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.
64. Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.
65. Determine whether the simplified expression is rational or irrational:
 $\sqrt{-18 - 4(5)(-1)}$.
66. Determine whether the simplified expression is rational or irrational:
 $\sqrt{-16 + 4(5) + 5}$.
67. The division of two natural numbers will always result in what type of number?
68. What property of real numbers would simplify the following expression:
 $4 + 7(x - 1)$?

3.2 working with roots

- 3.2.1 simplifying radical expressions (OpenStax College Algebra with Corequisite Support)**
- 3.2.2 rationalizing denominators/working with radical expressions (Modeling, Functions, and Graphs)**

59. Planck's constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck's constant in standard notation.

1.3 Radicals and Rational Exponents

Learning Objectives

In this section, you will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

COREQUISITE SKILLS

Learning Objective:

- Investigate the discipline called learning science and the idea of a knowledge space.

Objective 1: Investigate the discipline called learning science and the idea of a knowledge space.

The brain is a complex organ. It is the control center for our bodies, while the mind is where thinking and learning take place. In an attempt to understand the processes that occur in learning, researchers study a collection of disciplines called **learning sciences**. This interdisciplinary field includes study of psychological, sociological, anthropological, and computational approaches to learning.

In this skill sheet we will investigate the mathematics of mastery and knowledge spaces. A **knowledge space** includes the possible states of knowledge of a human learner. The theory of knowledge space was introduced in 1985 by mathematical psychologists Jean-Paul Doignon and Jean-Claude Falmagne and has since been studied by many researchers.¹

Practice Makes Perfect

Investigation: There are 32 student-learning outcomes (SLO's) in a typical College Algebra course. These are topics a student needs to master to show proficiency in College Algebra. Let's begin by looking at just a few of these skills. Let's assign the variables, A, B, C, and D to the following topics. We will name the set containing each of these 4 topics, Q.

- A = Graph the basic functions listed in the library of functions.
- B = Find the domain of a function defined by an equation.
- C = Create a new function through composition of functions.
- D = Find linear functions that model data sets.

Using roster notation $Q = \{A, B, C, D\}$.

1. List each of the possible **subsets** of the 4 topics listed above using roster notation. Remember a subset is a collection of topics in which each topic listed is an element of the set Q we defined above. By including a topic, we are indicating that the student has mastered the topic.
2. Verify in your work above you have listed all 16 subsets to set Q. Remember that a subset may contain all of the topics listed in Q.

1 Doignon, J.-P.; Falmagne, J.-Cl. (1985), "Spaces for the assessment of knowledge", International Journal of Man-Machine Studies.

3. What formula could you use to help you determine the number of possible subsets? Remember that each topic could be mastered or not by a student. Show below that your formula would be equal to 16 for a list of 4 topics.
4. Now use the formula you found in #3 to find the number of subsets possible if we include all 32 student-learning outcomes.
- Hint: In evaluating exponential terms, the function values increase very rapidly. To display very large (or very small) values, a calculator will use scientific notation. For example: 2.56 E6 is telling you to move the decimal point 6 places to the right and to insert zeros where you have missing values.
- For example: 2.56 E6 = 2,560,000 or 2 million, five hundred, sixty thousand.
5. The subsets you created in #1 are referred to as knowledge spaces in the field of learning science. In this context mastery of one concept may depend on your mastery of another.

List one skill in mathematics that would help to master each of the following SLO's:

- A = Graph the basic functions listed in the library of functions.
- B = Find the domain of a function defined by an equation.
- C = Create a new function through composition of functions.
- D = Find linear functions that model data sets.

6. Mastery of what are called **linchpin topics** will make it easier to learn other topics. For example, the ability to solve linear equations with variables on both sides can “unlock” a whole set of new skills for a student to master.

List 3 other linchpin topics that would help you to master this math course. Discuss these with others in your class. Did they identify the same topics?

- 1.
- 2.
- 3.

7. A corequisite course in mathematics is designed to provide support to a student by reviewing linchpin topics right when and where students need the help. Review of these important foundational ideas allow the learner to move on and master the student learning objectives for the course.

Brainstorm ideas with your classmates about ways this corequisite support course could help you in your learning.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in [Figure 1](#), and use the Pythagorean Theorem.

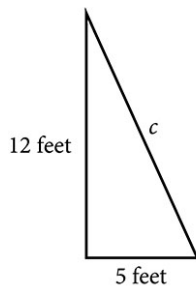


Figure 1

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 169 &= c^2 \end{aligned}$$

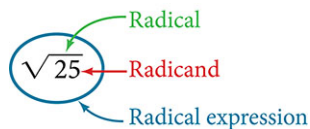
Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a . The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative number that when multiplied by itself equals a . The square root obtained using a calculator is the principal square root.

The principal square root of a is written as \sqrt{a} . The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.



Principal Square Root

The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a . It is written as a **radical expression**, with a symbol called a **radical** over the term called the **radicand**: \sqrt{a} .

Q&A Does $\sqrt{25} = \pm 5$?

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

EXAMPLE 1

Evaluating Square Roots

Evaluate each expression.

(a) $\sqrt{100}$ (b) $\sqrt{\sqrt{16}}$ (c) $\sqrt{25 + 144}$ (d) $\sqrt{49} - \sqrt{81}$

Solution

(a) $\sqrt{100} = 10$ because $10^2 = 100$ (b) $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$

(c) $\sqrt{25 + 144} = \sqrt{169} = 13$ because $13^2 = 169$ (d) $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Q&A For $\sqrt{25 + 144}$, can we find the square roots before adding?

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

TRY IT #1 Evaluate each expression.

(a) $\sqrt{225}$ (b) $\sqrt{\sqrt{81}}$ (c) $\sqrt{25 - 9}$ (d) $\sqrt{36} + \sqrt{121}$

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots*, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule for Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b .

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$



HOW TO

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

EXAMPLE 2

Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

(a) $\sqrt{300}$ (b) $\sqrt{162a^5b^4}$

✓ Solution

(a)

$$\sqrt{100 \cdot 3}$$

Factor perfect square from radicand.

$$\sqrt{100} \cdot \sqrt{3}$$

Write radical expression as product of radical expressions.

$$10\sqrt{3}$$

Simplify.

(b)

$$\sqrt{81a^4b^4 \cdot 2a}$$

Factor perfect square from radicand.

$$\sqrt{81a^4b^4} \cdot \sqrt{2a}$$

Write radical expression as product of radical expressions.

$$9a^2b^2\sqrt{2a}$$

Simplify.

> **TRY IT** #2 Simplify $\sqrt{50x^2y^3z}$.



HOW TO

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

EXAMPLE 3

Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.

$$\sqrt{12} \cdot \sqrt{3}$$

 **Solution** $\sqrt{12 \cdot 3}$ Express the product as a single radical expression. $\sqrt{36}$ Simplify.

6

 **TRY IT** #3 Simplify $\sqrt{50x} \cdot \sqrt{2x}$ assuming $x > 0$.

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots*. It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately.

We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

The Quotient Rule for Simplifying Square Roots

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b , where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

**HOW TO**

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.


EXAMPLE 4

Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$$\sqrt{\frac{5}{36}}$$

 **Solution**
 $\frac{\sqrt{5}}{\sqrt{36}}$ Write as quotient of two radical expressions.

 $\frac{\sqrt{5}}{6}$ Simplify denominator.
 **TRY IT** #4 Simplify $\sqrt{\frac{2x^2}{9y^4}}$.**EXAMPLE 5**

Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$$

 **Solution**

$$\sqrt{\frac{234x^{11}y}{26x^7y}}$$

Combine numerator and denominator into one radical expression.

$$\sqrt{9x^4}$$

Simplify fraction.

$$3x^2$$

Simplify square root.

 **TRY IT** #5 Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2} + \sqrt{18} = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$.



HOW TO

Given a radical expression requiring addition or subtraction of square roots, simplify.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

EXAMPLE 6


Adding Square Roots

Add $5\sqrt{12} + 2\sqrt{3}$.

 **Solution**

We can rewrite $5\sqrt{12}$ as $5\sqrt{4 \cdot 3}$. According to the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5(2)\sqrt{3}$, which is $10\sqrt{3}$. Now the terms have the same radicand so we can add.

$$10\sqrt{3} + 2\sqrt{3} = 12\sqrt{3}$$

 **TRY IT** #6 Add $\sqrt{5} + 6\sqrt{20}$.

EXAMPLE 7

Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c} - 14\sqrt{8a^3b^4c}$.

 **Solution**

Rewrite each term so they have equal radicands.

$$\begin{aligned} 20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\ &= 20(3)(2)|a|b^2\sqrt{2ac} \\ &= 120|a|b^2\sqrt{2ac} \end{aligned}$$

$$\begin{aligned} 14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\ &= 14(2)|a|b^2\sqrt{2ac} \\ &= 28|a|b^2\sqrt{2ac} \end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac} - 28|a|b^2\sqrt{2ac} = 92|a|b^2\sqrt{2ac}$$

 **TRY IT** #7 Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator*.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a + b\sqrt{c}$, then the conjugate is $a - b\sqrt{c}$.



HOW TO

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

- Multiply the numerator and denominator by the radical in the denominator.
- Simplify.

EXAMPLE 8

Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$\frac{2\sqrt{30}}{30}$$

$$\frac{\sqrt{30}}{15}$$

> **TRY IT** #8 Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.



HOW TO

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

EXAMPLE 9

Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1 + \sqrt{5}$ is $1 - \sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$\frac{4-4\sqrt{5}}{-4}$$

$$\sqrt{5} - 1$$

Use the distributive property.

Simplify.

> **TRY IT** #9 Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding n th Roots

Suppose we know that $a^3 = 8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3 = 8$, we say that 2 is the cube root of 8.

The n th root of a is a number that, when raised to the n th power, gives a . For example, -3 is the 5th root of -243 because $(-3)^5 = -243$. If a is a real number with at least one n th root, then the **principal n th root** of a is the number with the same sign as a that, when raised to the n th power, equals a .

The principal n th root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical

expression, n is called the **index** of the radical.

Principal n th Root

If a is a real number with at least one n th root, then the **principal n th root** of a , written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the n th power, equals a . The **index** of the radical is n .

EXAMPLE 10

Simplifying n th Roots

Simplify each of the following:

(a) $\sqrt[5]{-32}$ (b) $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$ (c) $-\sqrt[3]{\frac{8x^6}{125}}$ (d) $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

(a) $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

(b) First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$

(c) $\frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}}$ Write as quotient of two radical expressions.

$\frac{-2x^2}{5}$ Simplify.

(d) $8\sqrt[4]{3} - 2\sqrt[4]{3}$ Simplify to get equal radicands.

$6\sqrt[4]{3}$ Add.

 **TRY IT** #10 Simplify.

(a) $\sqrt[3]{-216}$ (b) $\frac{3\sqrt[4]{80}}{4\sqrt{5}}$ (c) $6\sqrt[3]{9,000} + 7\sqrt[3]{576}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an n th root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Rational Exponents

Rational exponents are another way to express principal n th roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$



HOW TO

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

EXAMPLE 11**Writing Rational Exponents as Radicals**

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

✓ **Solution**

The 2 tells us the power and the 3 tells us the root.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

> **TRY IT** #11 Write $9^{\frac{5}{2}}$ as a radical. Simplify.

EXAMPLE 12**Writing Radicals as Rational Exponents**

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

✓ **Solution**

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get

$$\frac{4}{\sqrt[7]{a^2}} = 4a^{-\frac{2}{7}}$$

> **TRY IT** #12 Write $x\sqrt{(5y)^9}$ using a rational exponent.

EXAMPLE 13**Simplifying Rational Exponents**

Simplify:

(a) $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$ (b) $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

✓ **Solution**

(a)


$30x^{\frac{3}{4}}x^{\frac{1}{5}}$ Multiply the coefficients.

$30x^{\frac{3}{4}+\frac{1}{5}}$ Use properties of exponents.

$30x^{\frac{19}{20}}$ Simplify.

(b)

$\left(\frac{9}{16}\right)^{\frac{1}{2}}$	Use definition of negative exponents.
$\sqrt{\frac{9}{16}}$	Rewrite as a radical.
$\frac{\sqrt{9}}{\sqrt{16}}$	Use the quotient rule.
$\frac{3}{4}$	Simplify.

 **TRY IT** #13 Simplify $(8x)^{\frac{1}{3}} \left(14x^{\frac{6}{5}}\right)$.

 **MEDIA**

Access these online resources for additional instruction and practice with radicals and rational exponents.

[Radicals \(http://openstax.org//introradical\)](http://openstax.org//introradical)

[Rational Exponents \(http://openstax.org//rationexpon\)](http://openstax.org//rationexpon)

[Simplify Radicals \(http://openstax.org//simpradical\)](http://openstax.org//simpradical)

[Rationalize Denominator \(http://openstax.org//rationdenom\)](http://openstax.org//rationdenom)



1.3 SECTION EXERCISES

Verbal

- What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.
- Where would radicals come in the order of operations? Explain why.
- Every number will have two square roots. What is the principal square root?
- Can a radical with a negative radicand have a real square root? Why or why not?

Numeric

For the following exercises, simplify each expression.

- | | | |
|------------------------------|-------------------------------|-------------------------------|
| 5. $\sqrt{256}$ | 6. $\sqrt{\sqrt{256}}$ | 7. $\sqrt{4(9+16)}$ |
| 8. $\sqrt{289} - \sqrt{121}$ | 9. $\sqrt{196}$ | 10. $\sqrt{1}$ |
| 11. $\sqrt{98}$ | 12. $\sqrt{\frac{27}{64}}$ | 13. $\sqrt{\frac{81}{5}}$ |
| 14. $\sqrt{800}$ | 15. $\sqrt{169} + \sqrt{144}$ | 16. $\sqrt{\frac{8}{50}}$ |
| 17. $\frac{18}{\sqrt{162}}$ | 18. $\sqrt{192}$ | 19. $14\sqrt{6} - 6\sqrt{24}$ |

20. $15\sqrt{5} + 7\sqrt{45}$

21. $\sqrt{150}$

22. $\sqrt{\frac{96}{100}}$

23. $(\sqrt{42})(\sqrt{30})$

24. $12\sqrt{3} - 4\sqrt{75}$

25. $\sqrt{\frac{4}{225}}$

26. $\sqrt{\frac{405}{324}}$

27. $\sqrt{\frac{360}{361}}$

28. $\frac{5}{1+\sqrt{3}}$

29. $\frac{8}{1-\sqrt{17}}$

30. $\sqrt[4]{16}$

31. $\sqrt[3]{128} + 3\sqrt[3]{2}$

32. $\sqrt[5]{\frac{-32}{243}}$

33. $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

34. $3\sqrt[3]{-432} + \sqrt[3]{16}$

Algebraic

For the following exercises, simplify each expression.

35. $\sqrt{400x^4}$

36. $\sqrt{4y^2}$

37. $\sqrt{49p}$

38. $(144p^2q^6)^{\frac{1}{2}}$

39. $m^{\frac{5}{2}}\sqrt{289}$

40. $9\sqrt{3m^2} + \sqrt{27}$

41. $3\sqrt{ab^2} - b\sqrt{a}$

42. $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

43. $\sqrt{\frac{225x^3}{49x}}$

44. $3\sqrt{44z} + \sqrt{99z}$

45. $\sqrt{50y^8}$

46. $\sqrt{490bc^2}$

47. $\sqrt{\frac{32}{14d}}$

48. $q^{\frac{3}{2}}\sqrt{63p}$

49. $\frac{\sqrt{8}}{1-\sqrt{3x}}$

50. $\sqrt{\frac{20}{121d^4}}$

51. $w^{\frac{3}{2}}\sqrt{32} - w^{\frac{3}{2}}\sqrt{50}$

52. $\sqrt{108x^4} + \sqrt{27x^4}$

53. $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

54. $\sqrt{147k^3}$

55. $\sqrt{125n^{10}}$

56. $\sqrt{\frac{42q}{36q^3}}$

57. $\sqrt{\frac{81m}{361m^2}}$

58. $\sqrt{72c} - 2\sqrt{2c}$

59. $\sqrt{\frac{144}{324d^2}}$

60. $\sqrt[3]{24x^6} + \sqrt[3]{81x^6}$

61. $\sqrt[4]{\frac{162x^6}{16x^4}}$

62. $\sqrt[3]{64y}$

63. $\sqrt[3]{128z^3} - \sqrt[3]{-16z^3}$

64. $\sqrt[5]{1,024c^{10}}$

Real-World Applications

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90,000 + 160,000}$. What is the length of the guy wire?
66. A car accelerates at a rate of $6 - \frac{\sqrt{4}}{\sqrt{t}}$ m/s² where t is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

67. $\frac{\sqrt{8}-\sqrt{16}}{4-\sqrt{2}} - 2\frac{1}{2}$

68. $\frac{4\frac{3}{2}-16\frac{3}{2}}{8\frac{1}{3}}$

69. $\frac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}} \cdot \frac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

70. $\frac{a}{a-\sqrt{c}}$

71. $\frac{x\sqrt{64y}+4\sqrt{y}}{\sqrt{128y}}$

72. $\left(\frac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\frac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\frac{\sqrt[3]{64}+\sqrt[4]{256}}{\sqrt{64}+\sqrt{256}}}$

1.4 Polynomials

Learning Objectives

In this section, you will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

COREQUISITE SKILLS

Learning Objectives

- Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Objective 1: Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Stanford University psychologist and researcher, Carol Dweck, PH.D., published a book in 2006 called *"Mindset, The New Psychology of Success"*, which changed how many people think about their talents and abilities. Based on decades of research Dr. Dweck outlined two mindsets and their influence on our learning.

Dr. Dweck's research found that people who believe that their abilities could change through learning and practice (**growth mindset**) more readily accepted learning challenges and persisted through these challenges. While individuals who believe that knowledge and abilities come from natural talent and cannot be changed (**fixed mindset**) more often become discouraged by failure and do not persist.

Her research shows that if we believe we can learn and master something new, this belief greatly improves our ability to

Rationalizing the Denominator

It is easier to work with radicals if there are no roots in the denominators of fractions. We can use the fundamental principle of fractions to remove radicals from the denominator. This process is called **rationalizing the denominator**. For square roots, we multiply the numerator and denominator of the fraction by the radical in the denominator.

Example A.90

Rationalize the denominator of each fraction.

a $\sqrt{\frac{1}{3}}$

b $\frac{\sqrt{2}}{\sqrt{50x}}$

Solution.

a Apply Property (2) to write the radical as a quotient.

$$\begin{aligned}\sqrt{\frac{1}{3}} &= \frac{\sqrt{1}}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} && \text{Multiply numerator and denominator by } \sqrt{3}. \\ &= \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

b It is always best to simplify the denominator before rationalizing.

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{50x}} &= \frac{\sqrt{2}}{5\sqrt{2x}} && \text{Multiply numerator and denominator by } \sqrt{2x}. \\ &= \frac{\sqrt{2} \cdot \sqrt{2x}}{5\sqrt{2x} \cdot \sqrt{2x}} && \text{Simplify.} \\ &= \frac{\sqrt{4x}}{5(2x)} \\ &= \frac{2\sqrt{x}}{10x} = \frac{\sqrt{x}}{5x}\end{aligned}$$

If the denominator of a fraction is a *binomial* in which one or both terms is a radical, we can use a special building factor to rationalize it. First, recall that

$$(p - q)(p + q) = p^2 - q^2$$

where the product consists of perfect squares only. Each of the two factors $p - q$ and $p + q$ is said to be the **conjugate** of the other.

Now consider a fraction of the form

$$\frac{a}{b + \sqrt{c}}$$

If we multiply the numerator and denominator of this fraction by the conjugate of the denominator, we get

$$\frac{a(b - \sqrt{c})}{(b + \sqrt{c})(b - \sqrt{c})} = \frac{ab - a\sqrt{c}}{b^2 - (\sqrt{c})^2} = \frac{ab - a\sqrt{c}}{b^2 - c}$$

The denominator of the fraction no longer contains any radicals—it has been rationalized.

Multiplying numerator and denominator by the conjugate of the denominator also works on fractions of the form

$$\frac{a}{\sqrt{b} + c} \quad \text{and} \quad \frac{a}{\sqrt{b} + \sqrt{c}}$$

We leave the verification of these cases as exercises.

Example A.91

Rationalize the denominator: $\frac{x}{\sqrt{2} + \sqrt{x}}$.

Solution. Multiply numerator and denominator by the conjugate of the denominator, $\sqrt{2} - \sqrt{x}$.

$$\frac{x(\sqrt{2} - \sqrt{x})}{(\sqrt{2} + \sqrt{x})(\sqrt{2} - \sqrt{x})} = \frac{x(\sqrt{2} - \sqrt{x})}{2 - x}$$

Simplifying $\sqrt[n]{x^n}$

Raising to a power is the inverse operation for extracting roots; that is,

$$(\sqrt[n]{a})^n = a$$

as long as $\sqrt[n]{a}$ is a real number. For example,

$$\left(\sqrt[4]{16}\right)^4 = 2^4 = 16, \quad \text{and} \quad \left(\sqrt[3]{-125}\right)^3 = (-5)^3 = -125$$

Now consider the power and root operations in the opposite order; is it true that $\sqrt[n]{a^n} = a$? If the index n is an odd number, then the statement is always true. For example,

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2 \quad \text{and} \quad \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$$

However, if n is even, we must be careful. Recall that the principal root $\sqrt[n]{x}$ is always positive, so if a is a negative number, it cannot be true that $\sqrt[n]{a^n} = a$. For example, if $a = -3$, then

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

Instead, we see that, for even roots, $\sqrt[n]{a^n} = |a|$.

We summarize our results in below.

Roots of Powers.

- | | |
|-------------------|-----------------------|
| 1 If n is odd, | $\sqrt[n]{a^n} = a$ |
| 2 If n is even, | $\sqrt[n]{a^n} = a $ |
| In particular, | $\sqrt{a^2} = a $ |

Example A.92

a $\sqrt{16x^2} = 4|x|$

b $\sqrt{(x-1)^2} = |x-1|$

SKILLS

Practice each skill in the exercises listed.

- 1 Simplify radicals: #1–6
- 2 Simplify products and quotients of radicals: #7–10
- 3 Combine like radicals: #11–18
- 4 Multiply radical expressions: #19–36
- 5 Rationalize the denominator: #37–50
- 6 Simplify $\sqrt[n]{a^n}$: #51–54
- 7 Solve radical equations: #55–80

Exercises A.10

For Problems 1-6, simplify. Assume that all variables represent positive numbers.

1.

a $\sqrt{18}$	b $\sqrt[3]{24}$	c $-\sqrt[4]{64}$
---------------	------------------	-------------------
2.

a $\sqrt{50}$	b $\sqrt[3]{54}$	c $-\sqrt[4]{162}$
---------------	------------------	--------------------
3.

a $\sqrt{60,000}$	b $\sqrt[3]{900,000}$	c $\sqrt[3]{\frac{-40}{27}}$
-------------------	-----------------------	------------------------------
4.

a $\sqrt{800,000}$	b $\sqrt[3]{24,000}$	c $\sqrt[4]{\frac{80}{625}}$
--------------------	----------------------	------------------------------
5.

a $\sqrt[3]{x^{10}}$	b $\sqrt{27z^3}$	c $\sqrt[4]{48a^9}$
----------------------	------------------	---------------------
6.

a $\sqrt[3]{y^{16}}$	b $\sqrt{12t^5}$	c $\sqrt[3]{81b^8}$
----------------------	------------------	---------------------

For Problems 7-10, simplify.

7.

a $-\sqrt{18s}\sqrt{2s^3}$	b $\sqrt[3]{7h^2}\sqrt[3]{-49h}$	c $\sqrt{16-4x^2}$
----------------------------	----------------------------------	--------------------
8.

a $\sqrt{3w^3}\sqrt{27w^3}$	b $-\sqrt[4]{2m^3}\sqrt[4]{8m}$	c $\sqrt{9Y^2+18}$
-----------------------------	---------------------------------	--------------------
9.

a $\sqrt[3]{8A^3+A^6}$	b $\frac{\sqrt{45x^3y^3}}{\sqrt{5y}}$	c $\frac{\sqrt[3]{8b^7}}{\sqrt[3]{a^6b^2}}$
------------------------	---------------------------------------	---
10.

a $\sqrt[3]{b^9-27b^3}$	b $\frac{\sqrt{98x^2y^3}}{\sqrt{xy}}$	c $\frac{\sqrt[3]{16r^4}}{\sqrt[3]{4t^3}}$
-------------------------	---------------------------------------	--

For Problems 11-18, simplify and combine like terms.

- | | |
|---|---|
| 11. $3\sqrt{7} + 2\sqrt{7}$ | 12. $5\sqrt{2} - 3\sqrt{2}$ |
| 13. $4\sqrt{3} - \sqrt{27}$ | 14. $\sqrt{75} + 2\sqrt{3}$ |
| 15. $\sqrt{50x} + \sqrt{32x}$ | 16. $\sqrt{8y} - \sqrt{18y}$ |
| 17. $3\sqrt[3]{16} - \sqrt[3]{2} - 2\sqrt[3]{54}$ | 18. $\sqrt[3]{81} + 2\sqrt[3]{24} - 3\sqrt[3]{3}$ |

For Problems 19-32, multiply.

19. $2(3 - \sqrt{5})$
21. $\sqrt{2}(\sqrt{6} + \sqrt{10})$
23. $\sqrt[3]{2}(\sqrt[3]{20} - 2\sqrt[3]{12})$
25. $(\sqrt{x} - 3)(\sqrt{x} + 3)$
27. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + 2\sqrt{3})$
29. $(\sqrt{5} - \sqrt{2})^2$
31. $(\sqrt{a} - 2\sqrt{b})^2$
20. $5(2 - \sqrt{7})$
22. $\sqrt{3}(\sqrt{12} - \sqrt{15})$
24. $\sqrt[3]{3}(2\sqrt[3]{18} + \sqrt[3]{36})$
26. $(2 + \sqrt{x})(2 - \sqrt{x})$
28. $(\sqrt{3} - \sqrt{5})(2\sqrt{3} + \sqrt{5})$
30. $(\sqrt{2} - 2\sqrt{3})^2$
32. $(\sqrt{2a} - 2\sqrt{b})(\sqrt{2a} + 2\sqrt{b})$

For Problems 33-36, verify by substitution that the number is a solution of the quadratic equation.

33. $x^2 - 2x - 2 = 0$, $1 + \sqrt{3}$
35. $x^2 + 6x - 9 = 0$, $-3 + 3\sqrt{2}$
34. $x^2 + 4x - 1 = 0$, $-2 + \sqrt{5}$
36. $4x^2 - 20x + 22 = 0$, $\frac{5 - \sqrt{3}}{2}$

For Problems 37-50, rationalize the denominator.

37. $\frac{6}{\sqrt{3}}$
40. $\sqrt{\frac{27x}{20}}$
43. $\frac{2\sqrt{3}}{\sqrt{2k}}$
46. $\frac{3}{7 - \sqrt{2}}$
49. $\frac{\sqrt{6} - 3}{2 - \sqrt{6}}$
38. $\frac{10}{\sqrt{5}}$
41. $\sqrt{\frac{2a}{b}}$
44. $\frac{6\sqrt{2}}{\sqrt{3v}}$
47. $\frac{x}{x - \sqrt{3}}$
50. $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$
39. $\sqrt{\frac{7x}{18}}$
42. $\sqrt{\frac{5p}{q}}$
45. $\frac{4}{1 + \sqrt{3}}$
48. $\frac{y}{\sqrt{5} - y}$

51. Use your calculator to graph each function, and explain the result.

a $y = \sqrt{x^2}$ b $y = \sqrt[3]{x^3}$

52. Use your calculator to graph each function, and explain the result.

a $y = (x^4)^{1/4}$ b $y = (x^5)^{1/5}$

For Problems 53-54, do not assume that variables represent positive numbers. Use absolute value bars as necessary to simplify the radicals.

53. a $\sqrt{4x^2}$ b $\sqrt{(x-5)^2}$ c $\sqrt{x^2 - 6x + 9}$
54. a $\sqrt{9x^2y^4}$ b $\sqrt{(2x-1)^2}$ c $\sqrt{9x^2 - 6x + 1}$

For Problems 55-78, solve

55. $\sqrt{x} - 5 = 3$
57. $\sqrt{y+6} = 2$
59. $4\sqrt{z} - 8 = -2$
61. $5 + 2\sqrt{6-2w} = 13$
63. $3z + 4 = \sqrt{3z+10}$
65. $2x + 1 = \sqrt{10x+5}$
67. $\sqrt{y+4} = y - 8$
56. $\sqrt{x} - 4 = 1$
58. $\sqrt{y-3} = 5$
60. $-3\sqrt{z} + 14 = 8$
62. $8 - 3\sqrt{9+2w} = -7$
64. $2x - 3 = \sqrt{7x-3}$
66. $4x + 5 = \sqrt{3x+4}$
68. $4\sqrt{x-4} = x$

3.3 polynomials

3.3.1 adding/subtracting/multiplying polynomials (OpenStax College Algebra with Corequisite Support)

3.3.2 factoring polynomials (OpenStax College Algebra with Corequisite Support)

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \cdot 1 \\ &= x \end{aligned}$$

The area of the front of the library can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get $4x^2 + \frac{3}{2}x - x$ ft², or $4x^2 + \frac{1}{2}x$ ft².

In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π , is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x - 9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.

Leading coefficient Degree

$$\underbrace{a_n x^n}_{\text{Leading term}} + \dots + a_2 x^2 + a_1 x + a_0$$

Polynomials

A **polynomial** is an expression that can be written in the form

$$a_n x^n + \dots + a_2 x^2 + a_1 x + a_0$$

Each real number a_i is called a **coefficient**. The number a_0 that is not multiplied by a variable is called a *constant*. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient**.



HOW TO

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.

3. Identify the coefficient of the leading term.

EXAMPLE 1

Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

(a) $3 + 2x^2 - 4x^3$ (b) $5t^5 - 2t^3 + 7t$ (c) $6p - p^3 - 2$

✓ Solution

(a) The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4 .

(b) The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

(c) The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$. The leading coefficient is the coefficient of that term, -1 .

> **TRY IT** #1 Identify the degree, leading term, and leading coefficient of the polynomial $4x^2 - x^6 + 2x - 6$.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.



HOW TO

Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

EXAMPLE 2

Adding Polynomials

Find the sum.

$$(12x^2 + 9x - 21) + (4x^3 + 8x^2 - 5x + 20)$$

✓ Solution

$$4x^3 + (12x^2 + 8x^2) + (9x - 5x) + (-21 + 20) \quad \text{Combine like terms.}$$

$$4x^3 + 20x^2 + 4x - 1 \quad \text{Simplify.}$$

⊙ Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

> **TRY IT** #2 Find the sum.

$$(2x^3 + 5x^2 - x + 1) + (2x^2 - 3x - 4)$$

EXAMPLE 3**Subtracting Polynomials**

Find the difference.

$$(7x^4 - x^2 + 6x + 1) - (5x^3 - 2x^2 + 3x + 2)$$

 **Solution**

$$7x^4 - x^2 + 6x + 1 - 5x^3 + 2x^2 - 2 \quad \text{Distribute negative sign.}$$

$$7x^4 - 5x^3 + x^2 + 6x - 3x + 1 - 2 \quad \text{Group like terms.}$$

$$7x^4 - 5x^3 + x^2 + 3x - 1 \quad \text{Combine/simplify.}$$

 **Analysis**

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

 **TRY IT** #3 Find the difference.

$$(-7x^3 - 7x^2 + 6x - 2) - (4x^3 - 6x^2 - x + 7)$$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the **FOIL** method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x + 7)$ to obtain the equivalent expression $2x + 14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

**HOW TO**

Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

EXAMPLE 4**Multiplying Polynomials Using the Distributive Property**

Find the product.

$$(2x + 1)(3x^2 - x + 4)$$

✔ **Solution**

$$2x(3x^2 - x + 4) + 1(3x^2 - x + 4) \quad \text{Use the distributive property.}$$

$$(6x^3 - 2x^2 + 8x) + (3x^2 - x + 4) \quad \text{Multiply.}$$

$$6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4 \quad \text{Combine like terms.}$$

$$6x^3 + x^2 + 7x + 4 \quad \text{Simplify.}$$

🔍 **Analysis**

We can use a table to keep track of our work, as shown in [Table 1](#). Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Table 1

> **TRY IT** #4 Find the product.

$$(3x + 2)(x^3 - 4x^2 + 7)$$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.

First terms Last terms

$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd$$

Inner terms
Outer terms

The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.



HOW TO

Given two binomials, use FOIL to simplify the expression.

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

EXAMPLE 5**Using FOIL to Multiply Binomials**

Use FOIL to find the product.

$$(2x - 18)(3x + 3)$$

 **Solution**

Find the product of the first terms.

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ \downarrow \quad \downarrow \\ 2x - 18 \quad 3x + 3 \end{array} & & 2x \cdot 3x = 6x^2 \end{array}$$

Find the product of the outer terms.

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ \downarrow \quad \downarrow \\ 2x - 18 \quad 3x + 3 \end{array} & & 2x \cdot 3 = 6x \end{array}$$

Find the product of the inner terms.

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ \downarrow \quad \downarrow \\ 2x - 18 \quad 3x + 3 \end{array} & & -18 \cdot 3x = -54x \end{array}$$

Find the product of the last terms.

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ \downarrow \quad \downarrow \\ 2x - 18 \quad 3x + 3 \end{array} & & -18 \cdot 3 = -54 \end{array}$$

$$6x^2 + 6x - 54x - 54 \quad \text{Add the products.}$$

$$6x^2 + (6x - 54x) - 54 \quad \text{Combine like terms.}$$

$$6x^2 - 48x - 54 \quad \text{Simplify.}$$

 **TRY IT** #5 Use FOIL to find the product.

$$(x + 7)(3x - 5)$$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a **perfect square trinomial**. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x + 5)^2 = x^2 + 10x + 25$$

$$(x - 3)^2 = x^2 - 6x + 9$$

$$(4x - 1)^2 = 16x^2 - 8x + 1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x + a)^2 = (x + a)(x + a) = x^2 + 2ax + a^2$$

**HOW TO****Given a binomial, square it using the formula for perfect square trinomials.**

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

EXAMPLE 6**Expanding Perfect Squares**Expand $(3x - 8)^2$. **Solution**

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2 - 2(3x)(8) + (-8)^2$$

Simplify.

$$9x^2 - 48x + 64.$$

TRY IT #6 Expand $(4x - 1)^2$.**Difference of Squares**Another special product is called the **difference of squares**, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x + 1)(x - 1)$ using the FOIL method.

$$\begin{aligned}(x + 1)(x - 1) &= x^2 - x + x - 1 \\ &= x^2 - 1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$\begin{aligned}(x + 5)(x - 5) &= x^2 - 25 \\ (x + 11)(x - 11) &= x^2 - 121 \\ (2x + 3)(2x - 3) &= 4x^2 - 9\end{aligned}$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q&A **Is there a special form for the sum of squares?***No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.***Difference of Squares**

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a + b)(a - b) = a^2 - b^2$$

**HOW TO**

Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

EXAMPLE 7**Multiplying Binomials Resulting in a Difference of Squares**

Multiply $(9x + 4)(9x - 4)$.

✓ Solution

Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

> TRY IT #7 Multiply $(2x + 7)(2x - 7)$.

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$(a + 2b)(4a - b - c)$	
$a(4a - b - c) + 2b(4a - b - c)$	Use the distributive property.
$4a^2 - ab - ac + 8ab - 2b^2 - 2bc$	Multiply.
$4a^2 + (-ab + 8ab) - ac - 2b^2 - 2bc$	Combine like terms.
$4a^2 + 7ab - ac - 2bc - 2b^2$	Simplify.

EXAMPLE 8**Multiplying Polynomials Containing Several Variables**

Multiply $(x + 4)(3x - 2y + 5)$.

✓ Solution

Follow the same steps that we used to multiply polynomials containing only one variable.

$x(3x - 2y + 5) + 4(3x - 2y + 5)$	Use the distributive property.
$3x^2 - 2xy + 5x + 12x - 8y + 20$	Multiply.
$3x^2 - 2xy + (5x + 12x) - 8y + 20$	Combine like terms.
$3x^2 - 2xy + 17x - 8y + 20$	Simplify.

> TRY IT #8 Multiply $(3x - 1)(2x + 7y - 9)$.

MEDIA

Access these online resources for additional instruction and practice with polynomials.

[Adding and Subtracting Polynomials \(http://openstax.org/l/addsubpoly\)](http://openstax.org/l/addsubpoly)

[Multiplying Polynomials \(http://openstax.org/l/multiplpoly\)](http://openstax.org/l/multiplpoly)

[Special Products of Polynomials \(http://openstax.org/l/specialpolyprod\)](http://openstax.org/l/specialpolyprod)



1.4 SECTION EXERCISES

Verbal

- Evaluate the following statement: The degree of a polynomial in standard form is the exponent of the leading term. Explain why the statement is true or false.
- Many times, multiplying two binomials with two variables results in a trinomial. This is not the case when there is a difference of two squares. Explain why the product in this case is also a binomial.
- You can multiply polynomials with any number of terms and any number of variables using four basic steps over and over until you reach the expanded polynomial. What are the four steps?
- State whether the following statement is true and explain why or why not: A trinomial is always a higher degree than a monomial.

Algebraic

For the following exercises, identify the degree of the polynomial.

- $7x - 2x^2 + 13$
- $14m^3 + m^2 - 16m + 8$
- $-625a^8 + 16b^4$
- $200p - 30p^2m + 40m^3$
- $x^2 + 4x + 4$
- $6y^4 - y^5 + 3y - 4$

For the following exercises, find the sum or difference.

- $(12x^2 + 3x) - (8x^2 - 19)$
- $(4z^3 + 8z^2 - z) + (-2z^2 + z + 6)$
- $(6w^2 + 24w + 24) - (3w^2 - 6w + 3)$
- $(7a^3 + 6a^2 - 4a - 13) + (-3a^3 - 4a^2 + 6a + 17)$
- $(11b^4 - 6b^3 + 18b^2 - 4b + 8) - (3b^3 + 6b^2 + 3b)$
- $(49p^2 - 25) + (16p^4 - 32p^2 + 16)$

For the following exercises, find the product.

- $(4x + 2)(6x - 4)$
- $(14c^2 + 4c)(2c^2 - 3c)$
- $(6b^2 - 6)(4b^2 - 4)$
- $(3d - 5)(2d + 9)$
- $(9v - 11)(11v - 9)$
- $(4t^2 + 7t)(-3t^2 + 4)$
- $(8n - 4)(n^2 + 9)$

For the following exercises, expand the binomial.

- $(4x + 5)^2$
- $(3y - 7)^2$
- $(12 - 4x)^2$
- $(4p + 9)^2$
- $(2m - 3)^2$
- $(3y - 6)^2$

30. $(9b + 1)^2$

For the following exercises, multiply the binomials.

31. $(4c + 1)(4c - 1)$

32. $(9a - 4)(9a + 4)$

33. $(15n - 6)(15n + 6)$

34. $(25b + 2)(25b - 2)$

35. $(4 + 4m)(4 - 4m)$

36. $(14p + 7)(14p - 7)$

37. $(11q - 10)(11q + 10)$

For the following exercises, multiply the polynomials.

38. $(2x^2 + 2x + 1)(4x - 1)$

39. $(4t^2 + t - 7)(4t^2 - 1)$

40. $(x - 1)(x^2 - 2x + 1)$

41. $(y - 2)(y^2 - 4y - 9)$

42. $(6k - 5)(6k^2 + 5k - 1)$

43. $(3p^2 + 2p - 10)(p - 1)$

44. $(4m - 13)(2m^2 - 7m + 9)$

45. $(a + b)(a - b)$

46. $(4x - 6y)(6x - 4y)$

47. $(4t - 5u)^2$

48. $(9m + 4n - 1)(2m + 8)$

49. $(4t - x)(t - x + 1)$

50. $(b^2 - 1)(a^2 + 2ab + b^2)$

51. $(4r - d)(6r + 7d)$

52. $(x + y)(x^2 - xy + y^2)$

Real-World Applications

53. A developer wants to purchase a plot of land to build a house. The area of the plot can be described by the following expression: $(4x + 1)(8x - 3)$ where x is measured in meters. Multiply the binomials to find the area of the plot in standard form.

54. A prospective buyer wants to know how much grain a specific silo can hold. The area of the floor of the silo is $(2x + 9)^2$. The height of the silo is $10x + 10$, where x is measured in feet. Expand the square and multiply by the height to find the expression that shows how much grain the silo can hold.

Extensions

For the following exercises, perform the given operations.

55. $(4t - 7)^2(2t + 1) - (4t^2 + 2t + 11)$

56. $(3b + 6)(3b - 6)(9b^2 - 36)$

57. $(a^2 + 4ac + 4c^2)(a^2 - 4c^2)$

1.5 Factoring Polynomials

Learning Objectives

In this section, you will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

expressed in factored form as $20x(3x - 2)$ units². We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20. The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Greatest Common Factor

The **greatest common factor** (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.



HOW TO

Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

EXAMPLE 1

Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3 , x^2 , and x is x . (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3 , y^2 , and y is y . Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy(2x^2y^2) = 6x^3y^3$, $3xy(15xy) = 45x^2y^2$, and $3xy(7) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2 + 15xy + 7)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3xy)(2x^2y^2 + 15xy + 7) = 6x^3y^3 + 45x^2y^2 + 21xy$.

TRY IT #1 Factor $x(b^2 - a) + 6(b^2 - a)$ by pulling out the GCF.

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b . The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

Factoring a Trinomial with Leading Coefficient 1

A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

 **Q&A** Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be prime.



HOW TO

Given a trinomial in the form $x^2 + bx + c$, factor it.

1. List factors of c .
2. Find p and q , a pair of factors of c with a sum of b .
3. Write the factored expression $(x + p)(x + q)$.

EXAMPLE 2

Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution

We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In the table below, we list factors until we find a pair with the desired sum.

Factors of -15	Sum of Factors
1, -15	-14
-1 , 15	14
3, -5	-2
-3 , 5	2

Now that we have identified p and q as -3 and 5, write the factored form as $(x - 3)(x + 5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x - 3)(x + 5) = x^2 + 2x - 15$.

 **Q&A** Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

 **TRY IT** #2 Factor $x^2 - 7x + 6$.

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2 + 5x + 3$ can be rewritten as $(2x + 3)(x + 1)$ using this process. We begin by rewriting the original expression as $2x^2 + 2x + 3x + 3$ and then factor each portion of the expression to obtain $2x(x + 1) + 3(x + 1)$. We then pull out the GCF of $(x + 1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form $ax^2 + bx + c$ by grouping, we find two numbers with a product of ac and a sum of b . We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.



HOW TO

Given a trinomial in the form $ax^2 + bx + c$, factor by grouping.

1. List factors of ac .
2. Find p and q , a pair of factors of ac with a sum of b .
3. Rewrite the original expression as $ax^2 + px + qx + c$.
4. Pull out the GCF of $ax^2 + px$.
5. Pull out the GCF of $qx + c$.
6. Factor out the GCF of the expression.

EXAMPLE 3

Factoring a Trinomial by Grouping

Factor $5x^2 + 7x - 6$ by grouping.

Solution

We have a trinomial with $a = 5$, $b = 7$, and $c = -6$. First, determine $ac = -30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factors until we find a pair with the desired sum.

Factors of -30	Sum of Factors
1, -30	-29
$-1, 30$	29
2, -15	-13
$-2, 15$	13
3, -10	-7
$-3, 10$	7

So $p = -3$ and $q = 10$.

$$5x^2 - 3x + 10x - 6$$

$$x(5x - 3) + 2(5x - 3)$$

$$(5x - 3)(x + 2)$$

Rewrite the original expression as $ax^2 + px + qx + c$.
Factor out the GCF of each part.
Factor out the GCF of the expression.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x - 3)(x + 2) = 5x^2 + 7x - 6$.

TRY IT #3 Factor

(a) $2x^2 + 9x + 9$ (b) $6x^2 + x - 1$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$



HOW TO

Given a perfect square trinomial, factor it into the square of a binomial.

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab .
3. Write the factored form as $(a + b)^2$.

EXAMPLE 4

Factoring a Perfect Square Trinomial

Factor $25x^2 + 20x + 4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2 = (5x)^2$ and $4 = 2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2) = 20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x + 2)^2$.

TRY IT #4 Factor $49x^2 - 14x + 1$.

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when

the two factors are multiplied.

$$a^2 - b^2 = (a + b)(a - b)$$

We can use this equation to factor any differences of squares.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a + b)(a - b)$$



HOW TO

Given a difference of squares, factor it into binomials.

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a + b)(a - b)$.

EXAMPLE 5

Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x + 5)(3x - 5)$.

 **TRY IT** #5 Factor $81y^2 - 100$.

 **Q&A** Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Similarly, the difference of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

$$x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$$

The sign of the first 2 is the *same* as the sign between $x^3 - 2^3$. The sign of the $2x$ term is *opposite* the sign between $x^3 - 2^3$. And the sign of the last term, 4, is *always positive*.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$



HOW TO

Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a + b)(a^2 - ab + b^2)$. For a difference of cubes, write the factored form as $(a - b)(a^2 + ab + b^2)$.

EXAMPLE 6

Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution

Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x + 8)(x^2 - 8x + 64)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

TRY IT #6 Factor the sum of cubes: $216a^3 + b^3$.

EXAMPLE 7

Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x - 5)(4x^2 + 10x + 25)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

TRY IT #7 Factor the difference of cubes: $1,000x^3 - 1$.

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}(2 + 5x^{\frac{1}{2}})$.

EXAMPLE 8**Factoring an Expression with Fractional or Negative Exponents**Factor $3x(x+2)^{-\frac{1}{3}} + 4(x+2)^{\frac{2}{3}}$. **Solution**Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.

$$(x+2)^{-\frac{1}{3}}(3x+4(x+2)) \quad \text{Factor out the GCF.}$$

$$(x+2)^{-\frac{1}{3}}(3x+4x+8) \quad \text{Simplify.}$$

$$(x+2)^{-\frac{1}{3}}(7x+8)$$

 **TRY IT** #8 Factor $2(5a-1)^{\frac{3}{4}} + 7a(5a-1)^{-\frac{1}{4}}$. **MEDIA**

Access these online resources for additional instruction and practice with factoring polynomials.

[Identify GCF \(<http://openstax.org//findgcfctofact>\)](http://openstax.org//findgcfctofact)[Factor Trinomials when a Equals 1 \(<http://openstax.org//facttrinom1>\)](http://openstax.org//facttrinom1)[Factor Trinomials when a is not equal to 1 \(<http://openstax.org//facttrinom2>\)](http://openstax.org//facttrinom2)[Factor Sum or Difference of Cubes \(<http://openstax.org//sumdifcube>\)](http://openstax.org//sumdifcube)**1.5 SECTION EXERCISES****Verbal**

- If the terms of a polynomial do not have a GCF, does that mean it is not factorable? Explain.
- A polynomial is factorable, but it is not a perfect square trinomial or a difference of two squares. Can you factor the polynomial without finding the GCF?
- How do you factor by grouping?

Algebraic*For the following exercises, find the greatest common factor.*

- $14x + 4xy - 18xy^2$
- $49mb^2 - 35m^2ba + 77ma^2$
- $30x^3y - 45x^2y^2 + 135xy^3$
- $200p^3m^3 - 30p^2m^3 + 40m^3$
- $36j^4k^2 - 18j^3k^3 + 54j^2k^4$
- $6y^4 - 2y^3 + 3y^2 - y$

For the following exercises, factor by grouping.

- $6x^2 + 5x - 4$
- $2a^2 + 9a - 18$
- $6c^2 + 41c + 63$
- $6n^2 - 19n - 11$
- $20w^2 - 47w + 24$
- $2p^2 - 5p - 7$

For the following exercises, factor the polynomial.

16. $7x^2 + 48x - 7$

17. $10h^2 - 9h - 9$

18. $2b^2 - 25b - 247$

19. $9d^2 - 73d + 8$

20. $90v^2 - 181v + 90$

21. $12t^2 + t - 13$

22. $2n^2 - n - 15$

23. $16x^2 - 100$

24. $25y^2 - 196$

25. $121p^2 - 169$

26. $4m^2 - 9$

27. $361d^2 - 81$

28. $324x^2 - 121$

29. $144b^2 - 25c^2$

30. $16a^2 - 8a + 1$

31. $49n^2 + 168n + 144$

32. $121x^2 - 88x + 16$

33. $225y^2 + 120y + 16$

34. $m^2 - 20m + 100$

35. $25p^2 - 120p + 144$

36. $36q^2 + 60q + 25$

For the following exercises, factor the polynomials.

37. $x^3 + 216$

38. $27y^3 - 8$

39. $125a^3 + 343$

40. $b^3 - 8d^3$

41. $64x^3 - 125$

42. $729q^3 + 1331$

43. $125r^3 + 1,728s^3$

44. $4x(x-1)^{-\frac{2}{3}} + 3(x-1)^{\frac{1}{3}}$

45. $3c(2c+3)^{-\frac{1}{4}} - 5(2c+3)^{\frac{3}{4}}$

46. $3t(10t+3)^{\frac{1}{3}} + 7(10t+3)^{\frac{4}{3}}$

47. $14x(x+2)^{-\frac{2}{5}} + 5(x+2)^{\frac{3}{5}}$

48. $9y(3y-13)^{\frac{1}{5}} - 2(3y-13)^{\frac{6}{5}}$

49. $5z(2z-9)^{-\frac{3}{2}} + 11(2z-9)^{-\frac{1}{2}}$

50. $6d(2d+3)^{-\frac{1}{6}} + 5(2d+3)^{\frac{5}{6}}$

Real-World Applications

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install statues and fountains in one of the city's parks. The park is a rectangle with an area of $98x^2 + 105x - 27$ m², as shown in the figure below. The length and width of the park are perfect factors of the area.

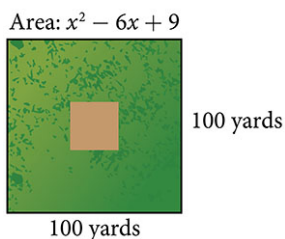


$$l \times w = 98x^2 + 105x - 27$$

51. Factor by grouping to find the length and width of the park.
52. A statue is to be placed in the center of the park. The area of the base of the statue is $4x^2 + 12x + 9\text{m}^2$. Factor the area to find the lengths of the sides of the statue.
53. At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2 - 25\text{m}^2$. Factor the area to find the lengths of the sides of the fountain.

For the following exercise, consider the following scenario:

A school is installing a flagpole in the central plaza. The plaza is a square with side length 100 yd. as shown in the figure below. The flagpole will take up a square plot with area $x^2 - 6x + 9\text{yd}^2$.



54. Find the length of the base of the flagpole by factoring.

Extensions

For the following exercises, factor the polynomials completely.

55. $16x^4 - 200x^2 + 625$ 56. $81y^4 - 256$ 57. $16z^4 - 2,401a^4$
58. $5x(3x + 2)^{-\frac{2}{4}} + (12x + 8)^{\frac{3}{2}}$ 59. $(32x^3 + 48x^2 - 162x - 243)^{-1}$

1.6 Rational Expressions

Learning Objectives

In this section, you will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

COREQUISITE SKILLS

Learning Objectives

- Identify the skills leading to successful preparation for a college level mathematics exam.
- Create a plan for success when taking mathematics exams.

Objective 1: Identify the skills leading to successful preparation for a college level mathematics exam.

Complete the following surveys by placing a checkmark in the a column for each strategy based on the frequency that you engaged in the strategy during your last academic term.

3.4 rational expressions (OpenStax College Algebra with Corequisite Support)

members.

A pastry shop has fixed costs of \$280 per week and variable costs of \$9 per box of pastries. The shop's costs per week in terms of x , the number of boxes made, is $280 + 9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x + 4)^2}{(x + 4)(x + 7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.

$$\frac{x + 4}{x + 7}$$



HOW TO

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

EXAMPLE 1

Simplifying Rational Expressions

Simplify $\frac{x^2 - 9}{x^2 + 4x + 3}$.

✓ Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)}$$

Factor the numerator and the denominator.

$$\frac{x-3}{x+1}$$

Cancel common factor $(x + 3)$.

🕒 Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.



Q&A Can the x^2 term be cancelled in **Example 1**?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.



TRY IT #1 Simplify $\frac{x-6}{x^2-36}$.

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.



HOW TO

Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

EXAMPLE 2

Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2 + 4x - 5}{3x + 18} \cdot \frac{2x - 1}{x + 5}$$

✓ Solution

$$\frac{(x+5)(x-1)}{3(x+6)} \cdot \frac{(2x-1)}{(x+5)}$$

Factor the numerator and denominator.

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$$

Multiply numerators and denominators.

$$\frac{\cancel{(x+5)}(x-1)(2x-1)}{3(x+6)\cancel{(x+5)}}$$

Cancel common factors to simplify.

$$\frac{(x-1)(2x-1)}{3(x+6)}$$

> **TRY IT** #2 Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2 + 11x + 30}{x^2 + 5x + 6} \cdot \frac{x^2 + 7x + 12}{x^2 + 8x + 16}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$



HOW TO

Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.

4. Multiply the denominators.
5. Simplify.

EXAMPLE 3**Dividing Rational Expressions**

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2 + x - 6}{x^2 - 1} \div \frac{x^2 - 4}{x^2 + 2x + 1}$$

 **Solution**

$$\frac{2x^2+x-6}{x^2-1} \cdot \frac{x^2+2x+1}{x^2-4}$$

Rewrite as multiplication.

$$\frac{(2x-3)(x+2)}{(x+1)(x-1)} \cdot \frac{(x+1)^2}{(x+2)(x-2)}$$

Factor.

$$\frac{(2x-3)(x+2)(x+1)^2}{(x+1)(x-1)(x+2)(x-2)}$$

Multiply.

$$\frac{(2x-3)(x+1)}{(x-1)(x-2)}$$

Cancel common factors to simplify.

 **TRY IT** #3 Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2 - 16}{3x^2 + 17x - 28} \div \frac{3x^2 - 2x - 8}{x^2 + 5x - 14}$$

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned} \frac{5}{24} + \frac{1}{40} &= \frac{25}{120} + \frac{3}{120} \\ &= \frac{28}{120} \\ &= \frac{7}{30} \end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x + 3)(x + 4)$ and $(x + 4)(x + 5)$, then the LCD would be $(x + 3)(x + 4)(x + 5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x + 3)(x + 4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x + 4)(x + 5)$ by $\frac{x+3}{x+3}$.

**HOW TO**

Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

EXAMPLE 4**Adding Rational Expressions**

Add the rational expressions:

$$\frac{5}{x} + \frac{6}{y}$$

✓ Solution

First, we have to find the LCD. In this case, the LCD will be xy . We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\begin{aligned} \frac{5}{x} \cdot \frac{y}{y} + \frac{6}{y} \cdot \frac{x}{x} \\ \frac{5y}{xy} + \frac{6x}{xy} \end{aligned}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x + 5y}{xy}$$

🕒 Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

EXAMPLE 5**Subtracting Rational Expressions**

Subtract the rational expressions:

$$\frac{6}{x^2 + 4x + 4} - \frac{2}{x^2 - 4}$$

✓ Solution

$$\begin{aligned} \frac{6}{(x+2)^2} - \frac{2}{(x+2)(x-2)} \\ \frac{6}{(x+2)^2} \cdot \frac{x-2}{x-2} - \frac{2}{(x+2)(x-2)} \cdot \frac{x+2}{x+2} \\ \frac{6(x-2)}{(x+2)^2(x-2)} - \frac{2(x+2)}{(x+2)^2(x-2)} \\ \frac{6x-12-(2x+4)}{(x+2)^2(x-2)} \\ \frac{4x-16}{(x+2)^2(x-2)} \\ \frac{4(x-4)}{(x+2)^2(x-2)} \end{aligned}$$

Factor.

Multiply each fraction to get LCD as denominator.

Multiply.

Apply distributive property.

Subtract.

Simplify.

Q&A Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

TRY IT #4 Subtract the rational expressions: $\frac{3}{x+5} - \frac{1}{x-3}$.

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the

numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1} \cdot \frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.



HOW TO

Given a complex rational expression, simplify it.

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

EXAMPLE 6

Simplifying Complex Rational Expressions

Simplify: $\frac{y + \frac{1}{x}}{\frac{x}{y}}$.

Solution

Begin by combining the expressions in the numerator into one expression.

$$y \cdot \frac{x}{x} + \frac{1}{x} \quad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator.}$$

$$\frac{xy}{x} + \frac{1}{x}$$

$$\frac{xy+1}{x}$$

Add numerators.

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$$\frac{xy+1}{x} \div \frac{x}{y}$$

$$\frac{xy+1}{x} \cdot \frac{y}{x}$$

$$\frac{y(xy+1)}{x^2}$$

Rewrite as multiplication.

Multiply.

TRY IT #5 Simplify: $\frac{\frac{x}{y} - \frac{y}{x}}{y}$

Q&A Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

MEDIA

Access these online resources for additional instruction and practice with rational expressions.

[Simplify Rational Expressions \(http://openstax.org//simpratexpress\)](http://openstax.org//simpratexpress)

[Multiply and Divide Rational Expressions \(http://openstax.org//multdivratex\)](http://openstax.org//multdivratex)

[Add and Subtract Rational Expressions \(http://openstax.org//addsubratex\)](http://openstax.org//addsubratex)

[Simplify a Complex Fraction \(http://openstax.org//complexfract\)](http://openstax.org//complexfract)



1.6 SECTION EXERCISES

Verbal

- How can you use factoring to simplify rational expressions?
- How do you use the LCD to combine two rational expressions?
- Tell whether the following statement is true or false and explain why: You only need to find the LCD when adding or subtracting rational expressions.

Algebraic

For the following exercises, simplify the rational expressions.

- $\frac{x^2-16}{x^2-5x+4}$
- $\frac{y^2+10y+25}{y^2+11y+30}$
- $\frac{6a^2-24a+24}{6a^2-24}$
- $\frac{9b^2+18b+9}{3b+3}$
- $\frac{m-12}{m^2-144}$
- $\frac{2x^2+7x-4}{4x^2+2x-2}$
- $\frac{6x^2+5x-4}{3x^2+19x+20}$
- $\frac{a^2+9a+18}{a^2+3a-18}$
- $\frac{3c^2+25c-18}{3c^2-23c+14}$
- $\frac{12n^2-29n-8}{28n^2-5n-3}$

For the following exercises, multiply the rational expressions and express the product in simplest form.

- $\frac{x^2-x-6}{2x^2+x-6} \cdot \frac{2x^2+7x-15}{x^2-9}$
- $\frac{c^2+2c-24}{c^2+12c+36} \cdot \frac{c^2-10c+24}{c^2-8c+16}$
- $\frac{2d^2+9d-35}{d^2+10d+21} \cdot \frac{3d^2+2d-21}{3d^2+14d-49}$
- $\frac{10h^2-9h-9}{2h^2-19h+24} \cdot \frac{h^2-16h+64}{5h^2-37h-24}$
- $\frac{6b^2+13b+6}{4b^2-9} \cdot \frac{6b^2+31b-30}{18b^2-3b-10}$
- $\frac{2d^2+15d+25}{4d^2-25} \cdot \frac{2d^2-15d+25}{25d^2-1}$
- $\frac{6x^2-5x-50}{15x^2-44x-20} \cdot \frac{20x^2-7x-6}{2x^2+9x+10}$
- $\frac{t^2-1}{t^2+4t+3} \cdot \frac{t^2+2t-15}{t^2-4t+3}$
- $\frac{2n^2-n-15}{6n^2+13n-5} \cdot \frac{12n^2-13n+3}{4n^2-15n+9}$
- $\frac{36x^2-25}{6x^2+65x+50} \cdot \frac{3x^2+32x+20}{18x^2+27x+10}$

For the following exercises, divide the rational expressions.

- $\frac{3y^2-7y-6}{2y^2-3y-9} \div \frac{y^2+y-2}{2y^2+y-3}$
- $\frac{6p^2+p-12}{8p^2+18p+9} \div \frac{6p^2-11p+4}{2p^2+11p-6}$
- $\frac{q^2-9}{q^2+6q+9} \div \frac{q^2-2q-3}{q^2+2q-3}$
- $\frac{18d^2+77d-18}{27d^2-15d+2} \div \frac{3d^2+29d-44}{9d^2-15d+4}$
- $\frac{16x^2+18x-55}{32x^2-36x-11} \div \frac{2x^2+17x+30}{4x^2+25x+6}$
- $\frac{144b^2-25}{72b^2-6b-10} \div \frac{18b^2-21b+5}{36b^2-18b-10}$
- $\frac{16a^2-24a+9}{4a^2+17a-15} \div \frac{16a^2-9}{4a^2+11a+6}$
- $\frac{22y^2+59y+10}{12y^2+28y-5} \div \frac{11y^2+46y+8}{24y^2-10y+1}$
- $\frac{9x^2+3x-20}{3x^2-7x+4} \div \frac{6x^2+4x-10}{x^2-2x+1}$

For the following exercises, add and subtract the rational expressions, and then simplify.

33. $\frac{4}{x} + \frac{10}{y}$

34. $\frac{12}{2q} - \frac{6}{3p}$

35. $\frac{4}{a+1} + \frac{5}{a-3}$

36. $\frac{c+2}{3} - \frac{c-4}{4}$

37. $\frac{y+3}{y-2} + \frac{y-3}{y+1}$

38. $\frac{x-1}{x+1} - \frac{2x+3}{2x+1}$

39. $\frac{3z}{z+1} + \frac{2z+5}{z-2}$

40. $\frac{4p}{p+1} - \frac{p+1}{4p}$

41. $\frac{x}{x+1} + \frac{y}{y+1}$

For the following exercises, simplify the rational expression.

42. $\frac{\frac{6}{y} - \frac{4}{x}}{y}$

43. $\frac{\frac{2}{a} + \frac{7}{b}}{b}$

44. $\frac{\frac{x}{4} - \frac{p}{8}}{p}$

45. $\frac{\frac{3}{a} + \frac{b}{6}}{\frac{2b}{3a}}$

46. $\frac{\frac{3}{x+1} + \frac{2}{x-1}}{\frac{x-1}{x+1}}$

47. $\frac{\frac{a}{b} - \frac{b}{a}}{\frac{a+b}{ab}}$

48. $\frac{\frac{2x}{3} + \frac{4x}{7}}{\frac{x}{2}}$

49. $\frac{\frac{2c}{c+2} + \frac{c-1}{c+1}}{\frac{2c+1}{c+1}}$

50. $\frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x}}$

Real-World Applications

51. Brenda is placing tile on her bathroom floor. The area of the floor is $15x^2 - 8x - 7$ ft². The area of one tile is $x^2 - 2x + 1$ ft². To find the number of tiles needed, simplify the rational expression:

$$\frac{15x^2 - 8x - 7}{x^2 - 2x + 1}$$



$$\text{Area} = 15x^2 - 8x - 7$$

52. The area of Lijuan's yard is $25x^2 - 625$ ft². A patch of sod has an area of $x^2 - 10x + 25$ ft². Divide the two areas and simplify to find how many pieces of sod Lijuan needs to cover her yard.

53. Elroi wants to mulch his garden. His garden is $x^2 + 18x + 81$ ft². One bag of mulch covers $x^2 - 81$ ft². Divide the expressions and simplify to find how many bags of mulch Elroi needs to mulch his garden.

Extensions

For the following exercises, perform the given operations and simplify.

54. $\frac{x^2+x-6}{x^2-2x-3} \cdot \frac{2x^2-3x-9}{x^2-x-2} \div \frac{10x^2+27x+18}{x^2+2x+1}$

55. $\frac{3y^2-10y+3}{3y^2+5y-2} \cdot \frac{2y^2-3y-20}{2y^2-y-15} \div \frac{y-4}{y-4}$

56. $\frac{\frac{4a+1}{2a-3} + \frac{2a-3}{2a+3}}{\frac{4a^2+9}{a}}$

57. $\frac{x^2+7x+12}{x^2+x-6} \div \frac{3x^2+19x+28}{8x^2-4x-24} \div \frac{2x^2+x-3}{3x^2+4x-7}$

3.5 big and small numbers (worksheet - Benjamin Kennedy)

Benjamin Kennedy

Big and Small Numbers

1 Introduction

It turns out that, for most of calculus to make sense, we need a good idea of which numbers are “bigger” and “smaller” than others. These are ideas that we have all been familiar with as long as we’ve been doing arithmetic; but we will need these ideas at the front of our minds, and we will need to be able to apply them in relatively abstract situations. Practicing with this is the point of this worksheet.

When we start to think about this, right away we run into a terminology issue. Here’s a question for you: *is -4 bigger than -5 ?*

Ordinarily, if someone says that a number x is “bigger” than a number y , they might mean either one of two things: that $x > y$ (that is, x is to the right of y on the number line), or that $|x| > |y|$ (that is, x is further from 0 than y is). For the purposes of this worksheet, let us agree on the following.

- Saying that x is *greater* than y means that $x > y$ — that is, x is to the right of y on the number line. Saying that x is *less* than y means that $x < y$ — that is, x is to the left of y on the number line.
- Saying that x is *bigger* than y means that $|x| > |y|$ — that is, x is further away from 0 than y is. Saying that x is *smaller* than y means that $|x| < |y|$ — that is, x is closer to 0 than y is.

The terms *greater than* and *less than* are standard in mathematics, and everybody agrees that these terms have the meanings we’ve just described. The terms *bigger than* and *smaller than* aren’t standardized in the same way.

So, to answer the above question (having agreed on terminology): -4 is greater than -5 , but -4 is also smaller than -5 . On the other hand, 4 is both less than 5 and smaller than 5 .

2 Size and reciprocals

Every nonzero number x has a reciprocal $\frac{1}{x}$. A number and its reciprocal always have the same sign. 1 and -1 are their own reciprocals. All other numbers have reciprocals of different sizes than the numbers themselves, and the way that x relates to $\frac{1}{x}$ depends on where x falls relative to -1 , 0 , and 1 . For instance,

if $x < -1$, then $-1 < \frac{1}{x} < 0$. In this case, $\frac{1}{x}$ is greater than x and smaller than x .

For example, the reciprocal of -5 is $\frac{1}{-5} = -0.2$.

QUESTION 1: Make statements like the above about the relationship between x and $\frac{1}{x}$ for the cases that $-1 < x < 0$, that $0 < x < 1$, and that $1 < x$. Then provide explicit examples illustrating each statement.

3 Size and powers

Suppose that x is a *positive* number, and that n is a positive whole number that is two or greater. The relation of x to x^n depends on where x falls relative to 1. For instance,

Suppose that x is a positive number and that n is a positive whole number that is two or greater. If $x > 1$, then x^n is greater than x and bigger than x .

For example, $(2.5)^3 = 15.625$.

QUESTION 2: Make a statement like the above about the relationship between x and x^n for the case that $0 < x < 1$, and provide an explicit example.

QUESTION 3: Make a statement like the above about the relationship between x and x^n for the case that $x = 1$, and provide an explicit example.

Thinking about x^n when $x < 0$ is *slightly* trickier because the sign of x^n changes depending on whether n is even or odd. For example, the first few powers of -2 are

$$(-2)^1 = -2, \quad (-2)^2 = 4, \quad (-2)^3 = -8, \quad (-2)^4 = 16.$$

Keeping this in mind, though, we can make similar statements about the relationship between x and x^n .

QUESTION 4: Complete the following statement, and then provide explicit examples illustrating the statement. *Suppose that $x < -1$ and that n is a positive whole number that is two or greater. Then, if n is odd, x^n If n is even,*

QUESTION 5: Complete the following statement, and then provide explicit examples illustrating the statement. *Suppose that $x = -1$ and that n is a positive whole number that is two or greater. Then, if n is odd, x^n If n is even,*

QUESTION 6: Complete the following statement, and then provide explicit examples illustrating the statement. *Suppose that $-1 < x < 0$ and that n is a positive whole number that is two or greater. Then x^n *

QUESTION 6: If x is a real number and n is a positive integer that is two or greater, what is the one possibility for x that we haven't yet discussed? Comment on the relationship between x and x^n in this case.

QUESTION 7: Describe exactly the real numbers x for which x^2 is *greater* than x . Describe exactly the real numbers x for which x^2 is *bigger* than x .

QUESTION 8: Describe exactly the real numbers x for which x^3 is *greater* than x . Describe exactly the real numbers x for which x^3 is *bigger* than x .

4 Size and fractions

If we have a fraction $\frac{a}{b}$, the following general slogan holds:

- Making the numerator *bigger* makes the whole fraction *bigger* [Example: $\frac{13}{7}$ is bigger than $\frac{3}{7}$];
- Making the numerator *smaller* makes the whole fraction *smaller* [Example: $\frac{-1}{5}$ is smaller than $\frac{-2}{5}$];
- Making the denominator *bigger* makes the whole fraction *smaller* [Example: $\frac{1}{3}$ is smaller than $\frac{1}{2}$];
- Making the denominator *smaller* makes the whole fraction *bigger* [Example: $\frac{2}{1}$ is bigger than $\frac{2}{21}$].

QUESTION 9: For each of the pairs of expressions below, say [with explanation] whether the left expression or the right expression is *bigger*, or whether they are equal, or whether you can't tell without more information. Also say [with explanation] whether the left expression or the right expression is *greater*, or whether they are equal, or whether you can't tell without more information.

$$(a) \quad \frac{x^2 + 2}{x^2 + 1} \quad \frac{x^2 + 1}{x^2 + 1}$$

$$(b) \quad \frac{x^2 + 2}{x^2 + 1} \quad \frac{x^2 + 2}{x^2 + 3}$$

$$(c) \quad \frac{x + 2}{x^2 + 1} \quad \frac{x + 1}{x^2 + 1}$$

$$(d) \quad \frac{-x^2 - 1}{x^2 + 4} \quad \frac{-x^2 - 1}{x^2 + 1}$$

5 “Really big” and “really small” numbers

There is no technical definition of “big” and “small” numbers, but it is really helpful to keep the following impressionistic ideas in mind.

- If x is a “small” number, $\frac{1}{x}$ is a “big” number (and vice-versa).
- If x and y are “small” numbers, then xy is “really small.”
- If x and y are “big” numbers, then xy is “really big.”

QUESTION 10: Suppose that x is a “big” number and that y is a “small” number. Following the “impressionistic ideas” described above, say whether each of the following numbers is “really small,” “small,” “big,” or “really big.” Give examples (with specific values for x and y) that illustrate your answers.

(a) x^2

(b) $\frac{1}{x^2}$

(c) $\frac{x}{y}$

(d) $\frac{y}{x+1}$



4 Topics for Derivatives

- 4.1 difference quotients and rational functions
 - 4.1.1 rational functions and key features (Active Prelude to Calculus)
 - 4.1.2 difference quotient/average rate of change (OpenStax Precalculus)
- 4.2 exponents and exponential functions
 - 4.2.1 simplifying expressions with exponents, negative exponents (OpenStax College Algebra with Corequisite Support)
 - 4.2.2 modeling with exponential functions (Active Prelude to Calculus)
- 4.3 linear and rational equations (OpenStax College Algebra with Corequisite Support)
 - 4.3.1 solving equations
 - 4.3.2 writing an equation of a line
 - 4.3.3 parallel and perpendicular lines
- 4.4 composition of functions
 - 4.4.1 composite functions (Active Prelude to Calculus)
 - 4.4.2 composition of functions and Chain Rule (TBIL)
 - 4.4.3 composite functions and differentiation strategies (TBIL)
- 4.5 solving quadratic equations by factoring (Modeling, Functions, and Graphs)
- 4.6 interval notation and solving inequalities (OpenStax College Algebra with Corequisite Support)

Calculus Fun Fact: A marine biologist studying the relationship between specific numbers of sea urchins and the depletion rate of nearby kelp plants would use calculus to find a relationship between the amounts of a quantity and the rate that quantity is changing.

4.1 difference quotients and rational functions

4.1.1 rational functions and key features (Active Prelude to Calculus)

4.1.2 difference quotient/average rate of change (OpenStax Precalculus)

5.4 Rational Functions

Motivating Questions

- What is a rational function?
 - How can we determine key information about a rational function from its algebraic structure?
 - Why are rational functions important?
-

The average rate of change of a function on an interval always involves a ratio. Indeed, for a given function f that interests us near $t = 2$, we can investigate its average rate of change on intervals near this value by considering

$$AV_{[2,2+h]} = \frac{f(2+h) - f(2)}{h}.$$

Suppose, for instance, that f measures the height of a falling ball at time t and is given by $f(t) = -16t^2 + 32t + 48$, which happens to be a polynomial function of degree 2. For this particular function, its average rate of change on $[1, 1+h]$ is

$$\begin{aligned} AV_{[2,2+h]} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{-16(2+h)^2 + 32(2+h) + 48 - (-16 \cdot 4 + 32 \cdot 2 + 48)}{h} \\ &= \frac{-64 - 64h - 16h^2 + 64 + 32h + 48 - (48)}{h} \\ &= \frac{-64h - 16h^2}{h}. \end{aligned}$$

Structurally, we observe that $AV_{[2,2+h]}$ is a ratio of the two functions $-64h - 16h^2$ and h . Moreover, both the numerator and the denominator of the expression are themselves polynomial functions of the variable h . Note that we may be especially interested in what occurs as $h \rightarrow 0$, as these values will tell us the average velocity of the moving ball on shorter and shorter time intervals starting at $t = 2$. At the same time, $AV_{[2,2+h]}$ is not defined for $h = 0$.

Ratios of polynomial functions arise in several different important circumstances. Sometimes we are interested in what happens when the denominator approaches 0, which makes the overall ratio undefined. In other situations, we may want to know what happens in the long term and thus consider what happens when the input variable increases without bound.

Preview Activity 5.4.1. A drug company¹ estimates that to produce a new drug, it will cost \$5 million in startup resources, and that once they reach production, each gram of the drug will cost \$2500 to make.

- Determine a formula for a function $C(q)$ that models the cost of producing q grams of the drug. What familiar kind of function is C ?

- b. The drug company needs to sell the drug at a price of more than \$2500 per gram in order to at least break even. To investigate how they might set prices, they first consider what their *average* cost per gram is. What is the *total* cost of producing 1000 grams? What is the *average* cost per gram to produce 1000 grams?
- c. What is the total cost of producing 10000 grams? What is the average cost per gram to produce 10000 grams?
- d. Our computations in (b) and (c) naturally lead us to define the “average cost per gram” function, $A(q)$, whose output is the average cost of producing q grams of the drug. What is a formula for $A(q)$?
- e. Explain why another formula for A is $A(q) = 2500 + \frac{5000000}{q}$.
- f. What can you say about the long-range behavior of A ? What does this behavior mean in the context of the problem?

5.4.1 Long-range behavior of rational functions

The functions $AV_{[2,2+h]} = \frac{-64h-16h^2}{h}$ and $A(q) = \frac{5000000+2500q}{q}$ are both examples of rational functions, since each is a ratio of polynomial functions. Formally, we have the following definition.

Definition 5.4.1 A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q . That is, r is rational provided that for some polynomial functions p and q , we have

$$r(x) = \frac{p(x)}{q(x)}.$$

◇

Like with polynomial functions, we are interested in such natural questions as

- What is the long range behavior of a given rational function?
- What is the domain of a given rational function?
- How can we determine where a given rational function’s value is 0?

We begin by focusing on the long-range behavior of rational functions. It’s important first to recall our earlier work with power functions of the form $p(x) = x^{-n}$ where $n = 1, 2, \dots$. For such functions, we know that $p(x) = \frac{1}{x^n}$ where $n > 0$ and that

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$$

since x^n increases without bound as $x \rightarrow \infty$. The same is true when $x \rightarrow -\infty$: $\lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0$. Thus, any time we encounter a quantity such as $\frac{1}{x^3}$, this quantity will approach 0 as x

¹This activity is based on p. 457ff in *Functions Modeling Change*, by Connally et al.

increases without bound, and this will also occur for any constant numerator. For instance,

$$\lim_{x \rightarrow \infty} \frac{2500}{x^2} = 0$$

since 2500 times a quantity approaching 0 will still approach 0 as x increases.

Activity 5.4.2. Consider the rational function $r(x) = \frac{3x^2 - 5x + 1}{7x^2 + 2x - 11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2 . In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound. In order to understand the long-range behavior of r , we choose to write the function in a different algebraic form.

- a. Note that we can multiply the formula for r by the form of 1 given by $1 = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$. Do so, and distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form.
- b. Having rewritten r , we are in a better position to evaluate $\lim_{x \rightarrow \infty} r(x)$. Using our work from (a), we have

$$\lim_{x \rightarrow \infty} r(x) = \lim_{x \rightarrow \infty} \frac{3 - \frac{5}{x} + \frac{1}{x^2}}{7 + \frac{2}{x} - \frac{11}{x^2}}$$

What is the exact value of this limit and why?

- c. Next, determine

$$\lim_{x \rightarrow -\infty} r(x) = \lim_{x \rightarrow -\infty} \frac{3 - \frac{5}{x} + \frac{1}{x^2}}{7 + \frac{2}{x} - \frac{11}{x^2}}$$

- d. Use *Desmos* to plot r on the interval $[-10, 10]$. In addition, plot the horizontal line $y = \frac{3}{7}$. What is the meaning of the limits you found in (b) and (c)?

Activity 5.4.3. Let $s(x) = \frac{3x - 5}{7x^2 + 2x - 11}$ and $u(x) = \frac{3x^2 - 5x + 1}{7x + 2}$. Note that both the numerator and denominator of each of these rational functions increases without bound as $x \rightarrow \infty$, and in addition that x^2 is the highest order term present in each of s and u .

- a. Using a similar algebraic approach to our work in Activity 5.4.2, multiply $s(x)$ by $1 = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$ and hence evaluate

$$\lim_{x \rightarrow \infty} \frac{3x - 5}{7x^2 + 2x - 11}$$

What value do you find?

- b. Plot the function $y = s(x)$ on the interval $[-10, 10]$. What is the graphical meaning of the limit you found in (a)?

- c. Next, use appropriate algebraic work to consider $u(x)$ and evaluate

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 5x + 1}{7x + 2}.$$

What do you find?

- d. Plot the function $y = u(x)$ on the interval $[-10, 10]$. What is the graphical meaning of the limit you computed in (c)?

We summarize and generalize the results of Activity 5.4.2 and Activity 5.4.3 as follows.

The long-term behavior of a rational function.

Let p and q be polynomial functions so that $r(x) = \frac{p(x)}{q(x)}$ is a rational function. Suppose that p has degree n with leading term $a_n x^n$ and q has degree m with leading term $b_m x^m$ for some nonzero constants a_n and b_m . There are three possibilities ($n < m$, $n = m$, and $n > m$) that result in three different behaviors of r :

- a. if $n < m$, then the degree of the numerator is less than the degree of the denominator, and thus

$$\lim_{n \rightarrow \infty} r(x) = \lim_{n \rightarrow \infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = 0,$$

which tells us that $y = 0$ is a horizontal asymptote of r ;

- b. if $n = m$, then the degree of the numerator equals the degree of the denominator, and thus

$$\lim_{n \rightarrow \infty} r(x) = \lim_{n \rightarrow \infty} \frac{a_n x^n + \cdots + a_0}{b_n x^n + \cdots + b_0} = \frac{a_n}{b_n},$$

which tells us that $y = \frac{a_n}{b_n}$ (the ratio of the coefficients of the highest order terms in p and q) is a horizontal asymptote of r ;

- c. if $n > m$, then the degree of the numerator is greater than the degree of the denominator, and thus

$$\lim_{n \rightarrow \infty} r(x) = \lim_{n \rightarrow \infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \pm\infty,$$

(where the sign of the limit depends on the signs of a_n and b_m) which tells us that r does not have a horizontal asymptote.

In both situations (a) and (b), the value of $\lim_{x \rightarrow -\infty} r(x)$ is identical to $\lim_{x \rightarrow \infty} r(x)$.

5.4.2 The domain of a rational function

Because a rational function can be written in the form $r(x) = \frac{p(x)}{q(x)}$ for some polynomial functions p and q , we have to be concerned about the possibility that a rational function's denominator is zero. Since polynomial functions always have their domain as the set of all real numbers, it follows that any rational function is only undefined at points where its denominator is zero.

The domain of a rational function.

Let p and q be polynomial functions so that $r(x) = \frac{p(x)}{q(x)}$ is a rational function. The domain of r is the set of all real numbers except those for which $q(x) = 0$.

Example 5.4.2 Determine the domain of the function $r(x) = \frac{5x^3+17x^2-9x+4}{2x^3-6x^2-8x}$.

Solution. To find the domain of any rational function, we need to determine where the denominator is zero. The best way to find these values exactly is to factor the denominator. Thus, we observe that

$$2x^3 - 6x^2 - 8x = 2x(x^2 - 3x - 4) = 2x(x + 1)(x - 4).$$

By the Zero Product Property, it follows that the denominator of r is zero at $x = 0$, $x = -1$, and $x = 4$. Hence, the domain of r is the set of all real numbers except -1 , 0 , and 4 . \square

We note that when it comes to determining the domain of a rational function, the numerator is irrelevant: all that matters is where the denominator is 0.

Activity 5.4.4. Determine the domain of each of the following functions. In each case, write a sentence to accurately describe the domain.

a. $f(x) = \frac{x^2 - 1}{x^2 + 1}$

d. $j(x) = \frac{(x + 5)(x - 3)(x + 1)(x - 4)}{(x + 1)(x + 3)(x - 5)}$

b. $g(x) = \frac{x^2 - 1}{x^2 + 3x - 4}$

e. $k(x) = \frac{2x^2 + 7}{3x^3 - 12x}$

c. $h(x) = \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x - 2}$

f. $m(x) = \frac{5x^2 - 45}{7(x - 2)(x - 3)^2(x^2 + 9)(x + 1)}$

5.4.3 Applications of rational functions

Rational functions arise naturally in the study of the average rate of change of a polynomial function, leading to expressions such as

$$AV_{[2,2+h]} = \frac{-64h - 16h^2}{h}.$$

We will study several subtle issues that correspond to such functions further in Section 5.5. For now, we will focus on a different setting in which rational functions play a key role.

In Section 5.3, we encountered a class of problems where a key quantity was modeled by a polynomial function. We found that if we considered a container such as a cylinder with fixed surface area, then the volume of the container could be written as a polynomial of a single variable. For instance, if we consider a circular cylinder with surface area 10 square feet, then we know that

$$S = 10 = 2\pi r^2 + 2\pi r h$$

and therefore $h = \frac{10 - 2\pi r^2}{2\pi r}$. Since the cylinder's volume is $V = \pi r^2 h$, it follows that

$$V = \pi r^2 h = \pi r^2 \left(\frac{10 - 2\pi r^2}{2\pi r} \right) = r(10 - 2\pi r^2),$$

which is a polynomial function of r .

What happens if we instead fix the volume of the container and ask about how surface area can be written as a function of a single variable?

Example 5.4.3 Suppose we want to construct a circular cylinder that holds 20 cubic feet of volume. How much material does it take to build the container? How can we state the amount of material as a function of a single variable?

Solution. Neglecting any scrap, the amount of material it takes to construct the container is its surface area, which we know to be

$$S = 2\pi r^2 + 2\pi r h.$$

Because we want the volume to be fixed, this results in a constraint equation that enables us to relate r and h . In particular, since

$$V = 20 = \pi r^2 h,$$

it follows that we can solve for h and get $h = \frac{20}{\pi r^2}$. Substituting this expression for h in the equation for surface area, we find that

$$S = 2\pi r^2 + 2\pi r \cdot \frac{20}{\pi r^2} = 2\pi r^2 + \frac{40}{r}.$$

Getting a common denominator, we can also write S in the form

$$S(r) = \frac{2\pi r^3 + 40}{r}$$

and thus we see that S is a rational function of r . Because of the physical context of the problem and the fact that the denominator of S is r , the domain of S is the set of all positive real numbers. \square

Activity 5.4.5. Suppose that we want to build an open rectangular box (that is, without a top) that holds 15 cubic feet of volume. If we want one side of the base to be twice as long as the other, how does the amount of material required depend on the shorter side of the base? We investigate this question through the following sequence

of prompts.

- Draw a labeled picture of the box. Let x represent the shorter side of the base and h the height of the box. What is the length of the longer side of the base in terms of x ?
- Use the given volume constraint to write an equation that relates x and h , and solve the equation for h in terms of x .
- Determine a formula for the surface area, S , of the box in terms of x and h .
- Using the constraint equation from (b) together with your work in (c), write surface area, S , as a function of the single variable x .
- What type of function is S ? What is its domain?
- Plot the function S using *Desmos*. What appears to be the least amount of material that can be used to construct the desired box that holds 15 cubic feet of volume?

5.4.4 Summary

- A rational function is a function whose formula can be written as the ratio of two polynomial functions. For instance, $r(x) = \frac{7x^3 - 5x + 16}{-4x^4 + 2x^3 - 11x + 3}$ is a rational function.
- Two aspects of rational functions are straightforward to determine for any rational function. Given $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials, the domain of r is the set of all real numbers except any values of x for which $q(x) = 0$. In addition, we can determine the long-range behavior of r by examining the highest order terms in p and q :
 - if the degree of p is less than the degree of q , then r has a horizontal asymptote at $y = 0$;
 - if the degree of p equals the degree of q , then r has a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of p and q respectively;
 - and if the degree of p is greater than the degree of q , then r does not have a horizontal asymptote.
- Two reasons that rational functions are important are that they arise naturally when we consider the average rate of change on an interval whose length varies and when we consider problems that relate the volume and surface area of three-dimensional containers when one of those two quantities is constrained.

5.4.5 Exercises



1. Find the horizontal asymptote, if it exists, of the rational function below.

$$g(x) = \frac{(-1-x)(-7-2x)}{2x^2+1}$$



2. Compare and discuss the long-run behaviors of the functions below. In each blank, enter either the constant or the polynomial that the rational function behaves like as $x \rightarrow \pm\infty$:

$$f(x) = \frac{x^3+3}{x^3-8}, g(x) = \frac{x^2+3}{x^3-8}, \text{ and } h(x) = \frac{x^4+3}{x^3-8}$$

$f(x)$ will behave like the function $y = \underline{\hspace{2cm}}$ as $x \rightarrow \pm\infty$.

$g(x)$ will behave like the function $y = \underline{\hspace{2cm}}$ as $x \rightarrow \pm\infty$.

$h(x)$ will behave like the function $y = \underline{\hspace{2cm}}$ as $x \rightarrow \pm\infty$.



3. Let $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials of degrees m and n respectively.

(a) If $r(x) \rightarrow 0$ as $x \rightarrow \infty$, then

$m > n$ $m = n$ $m < n$ None of the above

(b) If $r(x) \rightarrow k$ as $x \rightarrow \infty$, with $k \neq 0$, then

$m < n$ $m > n$ $m = n$ None of the above



4. Find all zeros and vertical asymptotes of the rational function

$$f(x) = \frac{x+6}{(x+9)^2}$$

(a) The function has zero(s) at $x = \underline{\hspace{2cm}}$

(b) The function has vertical asymptote(s) at $x = \underline{\hspace{2cm}}$

(c) The function's long-run behavior is that $y \rightarrow \underline{\hspace{2cm}}$ as $x \rightarrow \pm\infty$

(d) On a piece of paper, sketch a graph of this function without using your calculator.



5. Find all zeros and vertical asymptotes of the rational function

$$f(x) = \frac{x^2-16}{-x^3-16x^2}$$

(a) The function has x-intercept(s) at $x = \underline{\hspace{2cm}}$

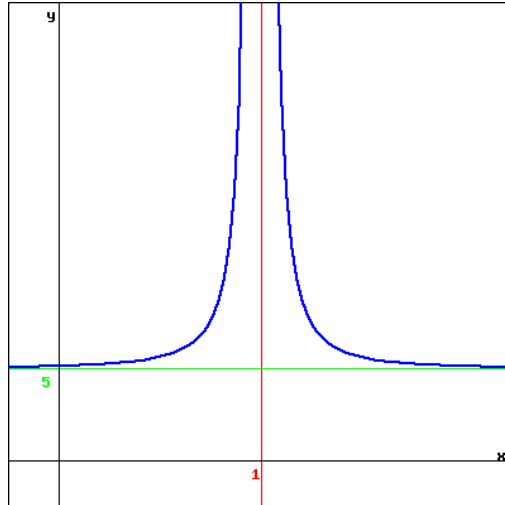
(b) The function has y-intercept(s) at $y = \underline{\hspace{2cm}}$

(c) The function has vertical asymptote(s) when $x = \underline{\hspace{2cm}}$

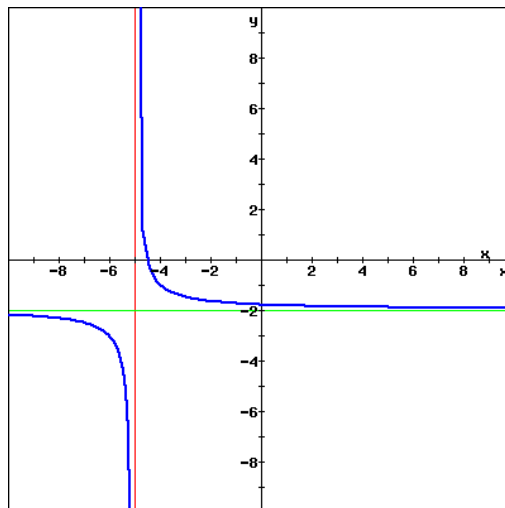
(d) The function has horizontal asymptote(s) when $y = \underline{\hspace{2cm}}$



6. Using the graph of the rational function $y = f(x)$ given in the figure below, evaluate the limits.



- (a) $\lim_{x \rightarrow \infty} f(x)$
- (b) $\lim_{x \rightarrow -\infty} f(x)$
- (c) $\lim_{x \rightarrow 1^+} f(x)$
- (d) $\lim_{x \rightarrow 1^-} f(x)$
7. The graph below is a vertical and/or horizontal shift of $y = 1/x$ (assume no reflections or compression/expansions have been applied).



- (a) The graph's equation can be written in the form

$$f(x) = \frac{1}{x + A} + B$$

for constants A and B . Based on the graph above, find the values for A and B .

- (b) Now take your formula from part (a) and write it as the ratio of two linear polyno-

mials of the form,

$$f(x) = \frac{Mx + C}{x + D}$$

for constants M , C , and D . What are the values of M , C , and D ?

(c) Find the exact values of the coordinates of the x - and y -intercepts of the graph.



8. Find all zeros and vertical asymptotes of the rational function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

- (a) The function has x -intercept(s) at $x =$ _____
 (b) The function has y -intercept(s) at $y =$ _____
 (c) The function has vertical asymptote(s) when $x =$ _____
 (d) The function has horizontal asymptote(s) when $y =$ _____

9. For each rational function below, determine the function's domain as well as the exact value of any horizontal asymptote.

a. $f(x) = \frac{17x^2 + 34}{19x^2 - 76}$

b. $g(x) = \frac{29}{53} + \frac{1}{x - 2}$

c. $h(x) = \frac{4 - 31x}{11x - 7}$

d. $r(x) = \frac{151(x - 4)(x + 5)^2(x - 2)}{537(x + 5)(x + 1)(x^2 + 1)(x - 15)}$

10. A rectangular box is being constructed so that its base is 1.5 times as long as it is wide. In addition, suppose that material for the base and top of the box costs \$3.75 per square foot, while material for the sides costs \$2.50 per square foot. Finally, we want the box to hold 8 cubic feet of volume.

- a. Draw a labeled picture of the box with x as the length of the shorter side of the box's base and h as its height.
 b. Determine a formula involving x and h for the total surface area, S , of the box.
 c. Use your work from (b) along with the given information about cost to determine a formula for the total cost, C , of the box in terms of x and h .
 d. Use the volume constraint given in the problem to write an equation that relates x and h , and solve that equation for h in terms of x .
 e. Combine your work in (c) and (d) to write the cost, C , of the box as a function solely of x .
 f. What is the domain of the cost function? How does a graph of the cost function appear? What does this suggest about the ideal box for the given constraints?

Other device ownership and usage trends may go in different directions by generation. From 2018 to 2019, Millennial tablet computer ownership dropped from 64% to 52%. But during the same period, the Baby Boom generation's tablet computer ownership stayed exactly even with 52% reporting ownership. And the 74-and-older group's tablet ownership increased from 25% to 33%.²

What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the slope of the curve at that point is close to the slope of the tangent line at that point.

Figure 1 represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
- slope at $x = -1$ is -1
- slope at $x = 2$ is 8

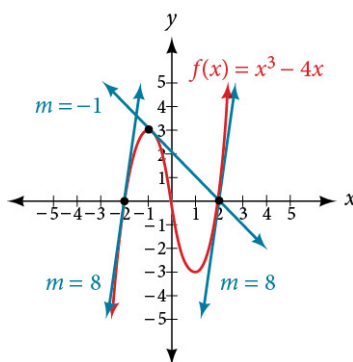


Figure 1 Graph showing tangents to curve at -2, -1, and 2.

Let's imagine a point on the curve of function f at $x = a$ as shown in **Figure 2**. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an x -value increased by some small real number h . The coordinates of this second point are $(a + h, f(a + h))$ for some positive-value h .

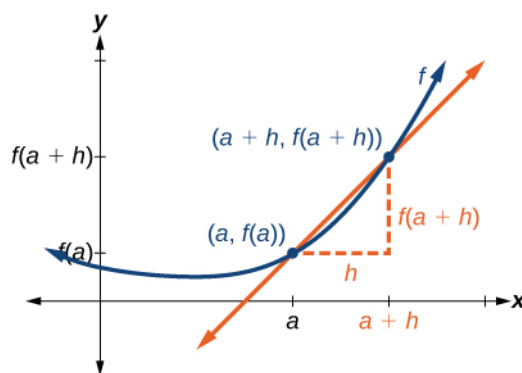


Figure 2 Connecting point a with a point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a + h, f(a + h))$, called a **secant line**, by applying the slope formula,

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

² <https://www.pewresearch.org/fact-tank/2019/09/09/us-generations-technology-use/>

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$\begin{aligned} m_{\text{sec}} &= \frac{f(a+h)-f(a)}{(a+h)-(a)} \\ &= \frac{f(a+h)-f(a)}{h} \end{aligned}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a+h, f(a+h))$.

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}$$

The Average Rate of Change between Two Points on a Curve

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h}$$

EXAMPLE 1

Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution

We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h}$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1 , which equals $-1 - 2 = -3$.

For the other point, $f(a+h)$ is the y -coordinate at $a+h$, which is $2 + (-3)$ or -1 , so $f(a+h) = f(-1) = 5$.

$$\begin{aligned} \text{AROC} &= \frac{f(a+h)-f(a)}{h} \\ &= \frac{5-(-6)}{-3} \\ &= \frac{11}{-3} \\ &= -\frac{11}{3} \end{aligned}$$

 **TRY IT** #1 Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$.

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in [Figure 2](#) smaller and smaller. Then $a+h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a+h, f(a+h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$. See [Figure 3](#).

4.2 exponents and exponential functions

- 4.2.1 simplifying expressions with exponents, negative exponents (OpenStax College Algebra with Corequisite Support)
- 4.2.2 modeling with exponential functions (Active Prelude to Calculus)

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
9-9:30pm							
9:30-10pm							
10-10:30pm							
10:30-11pm							
11-11:30pm							

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, x , but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \cdot x^4 &= \overset{3 \text{ factors}}{x \cdot x \cdot x} \cdot \overset{4 \text{ factors}}{x \cdot x \cdot x \cdot x} \\ &= \overset{7 \text{ factors}}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents*.

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n , the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

EXAMPLE 1**Using the Product Rule**

Write each of the following products with a single base. Do not simplify further.

(a) $t^5 \cdot t^3$ (b) $(-3)^5 \cdot (-3)$ (c) $x^2 \cdot x^5 \cdot x^3$

✓ **Solution**

Use the product rule to simplify each expression.

(a) $t^5 \cdot t^3 = t^{5+3} = t^8$ (b) $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$ (c) $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

> **TRY IT** #1 Write each of the following products with a single base. Do not simplify further.

(a) $k^6 \cdot k^9$ (b) $(\frac{2}{y})^4 \cdot (\frac{2}{y})$ (c) $t^3 \cdot t^6 \cdot t^5$

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$.

Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned} \frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot y \cdot y \cdot y \cdot y}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4 \end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

The Quotient Rule of Exponents

For any real number a and natural numbers m and n , such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

EXAMPLE 2**Using the Quotient Rule**

Write each of the following products with a single base. Do not simplify further.

(a) $\frac{(-2)^{14}}{(-2)^9}$ (b) $\frac{t^{23}}{t^{15}}$ (c) $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

✓ **Solution**

Use the quotient rule to simplify each expression.

(a) $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$ (b) $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$ (c) $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

> **TRY IT** #2 Write each of the following products with a single base. Do not simplify further.

(a) $\frac{s^{75}}{s^{68}}$ (b) $\frac{(-3)^6}{-3}$ (c) $\frac{(ef^2)^5}{(ef^2)^3}$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression $(x^2)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}(x^2)^3 &= \overset{3 \text{ factors}}{(x^2) \cdot (x^2) \cdot (x^2)} \\ &= \left(\overset{2 \text{ factors}}{x \cdot x}\right) \cdot \left(\overset{2 \text{ factors}}{x \cdot x}\right) \cdot \left(\overset{2 \text{ factors}}{x \cdot x}\right) \\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $(x^2)^3 = x^{2 \cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m \cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

Product Rule		Power Rule
$5^3 \cdot 5^4 = 5^{3+4}$	$= 5^7$	but $(5^3)^4 = 5^{3 \cdot 4} = 5^{12}$
$x^5 \cdot x^2 = x^{5+2}$	$= x^7$	but $(x^5)^2 = x^{5 \cdot 2} = x^{10}$
$(3a)^7 \cdot (3a)^{10} = (3a)^{7+10}$	$= (3a)^{17}$	but $((3a)^7)^{10} = (3a)^{7 \cdot 10} = (3a)^{70}$

The Power Rule of Exponents

For any real number a and positive integers m and n , the power rule of exponents states that

$$(a^m)^n = a^{m \cdot n}$$

EXAMPLE 3**Using the Power Rule**

Write each of the following products with a single base. Do not simplify further.

Ⓐ $(x^2)^7$ Ⓑ $((2t)^5)^3$ Ⓒ $((-3)^5)^{11}$

✓ **Solution**

Use the power rule to simplify each expression.

Ⓐ $(x^2)^7 = x^{2 \cdot 7} = x^{14}$ Ⓑ $((2t)^5)^3 = (2t)^{5 \cdot 3} = (2t)^{15}$ Ⓒ $((-3)^5)^{11} = (-3)^{5 \cdot 11} = (-3)^{55}$

> **TRY IT** #3 Write each of the following products with a single base. Do not simplify further.

Ⓐ $((3y)^8)^3$ Ⓑ $(t^5)^7$ Ⓒ $((-g)^4)^4$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0 . This appears later in more advanced courses, but for now, we will consider the value to be undefined.

The Zero Exponent Rule of Exponents

For any nonzero real number a , the zero exponent rule of exponents states that

$$a^0 = 1$$

EXAMPLE 4**Using the Zero Exponent Rule**

Simplify each expression using the zero exponent rule of exponents.

Ⓐ $\frac{c^3}{c^3}$ Ⓑ $\frac{-3x^5}{x^5}$ Ⓒ $\frac{(j^2k)^4}{(j^2k) \cdot (j^2k)^3}$ Ⓓ $\frac{5(rs^2)^2}{(rs^2)^2}$

✓ **Solution**

Use the zero exponent and other rules to simplify each expression.

Ⓐ

$$\begin{aligned} \frac{c^3}{c^3} &= c^{3-3} \\ &= c^0 \\ &= 1 \end{aligned}$$

$$\begin{aligned}
 \textcircled{b} \quad \frac{-3x^5}{x^5} &= -3 \cdot \frac{x^5}{x^5} \\
 &= -3 \cdot x^{5-5} \\
 &= -3 \cdot x^0 \\
 &= -3 \cdot 1 \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{c} \quad \frac{(j^2k)^4}{(j^2k) \cdot (j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator.} \\
 &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify.} \\
 &= (j^2k)^{4-4} && \text{Use the quotient rule.} \\
 &= (j^2k)^0 && \text{Simplify.} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \quad \frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule.} \\
 &= 5(rs^2)^0 && \text{Simplify.} \\
 &= 5 \cdot 1 && \text{Use the zero exponent rule.} \\
 &= 5 && \text{Simplify.}
 \end{aligned}$$

TRY IT #4 Simplify each expression using the zero exponent rule of exponents.

$$\textcircled{a} \quad \frac{i^7}{i^7} \quad \textcircled{b} \quad \frac{(de^2)^{11}}{2(de^2)^{11}} \quad \textcircled{c} \quad \frac{w^4 \cdot w^2}{w^6} \quad \textcircled{d} \quad \frac{i^3 \cdot i^4}{i^2 \cdot i^5}$$

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\begin{aligned}
 \frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\
 &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\
 &= \frac{1}{h \cdot h} \\
 &= \frac{1}{h^2}
 \end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}
 \frac{h^3}{h^5} &= h^{3-5} \\
 &= h^{-2}
 \end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing

a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n . Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n .

The Negative Rule of Exponents

For any nonzero real number a and natural number n , the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE 5

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

(a) $\frac{\theta^3}{\theta^{10}}$ (b) $\frac{z^2 \cdot z}{z^4}$ (c) $\frac{(-5r^3)^4}{(-5r^3)^8}$

✓ **Solution**

(a) $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$ (b) $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$

(c) $\frac{(-5r^3)^4}{(-5r^3)^8} = (-5r^3)^{4-8} = (-5r^3)^{-4} = \frac{1}{(-5r^3)^4}$

> **TRY IT** #5 Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

(a) $\frac{(-3t)^2}{(-3t)^8}$ (b) $\frac{f^{47}}{f^{49} \cdot f}$ (c) $\frac{2k^4}{5k^7}$

EXAMPLE 6

Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

(a) $b^2 \cdot b^{-8}$ (b) $(-x)^5 \cdot (-x)^{-5}$ (c) $\frac{-7z}{(-7z)^5}$

✓ **Solution**

(a) $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$ (b) $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$

(c) $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

> **TRY IT** #6 Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

(a) $t^{-11} \cdot t^6$ (b) $\frac{25^{12}}{25^{13}}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents*,

which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= \overset{3 \text{ factors}}{(pq) \cdot (pq) \cdot (pq)} \\ &= p \cdot q \cdot p \cdot q \cdot p \cdot q \\ &= \overset{3 \text{ factors}}{p \cdot p \cdot p} \cdot \overset{3 \text{ factors}}{q \cdot q \cdot q} \\ &= p^3 \cdot q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

The Power of a Product Rule of Exponents

For any real numbers a and b and any integer n , the power of a product rule of exponents states that

$$(ab)^n = a^n b^n$$

EXAMPLE 7

Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

Ⓐ $(ab^2)^3$ Ⓑ $(2t)^{15}$ Ⓒ $(-2w^3)^3$ Ⓓ $\frac{1}{(-7z)^4}$ Ⓔ $(e^{-2}f^2)^7$

✓ Solution

Use the product and quotient rules and the new definitions to simplify each expression.

Ⓐ $(ab^2)^3 = (a)^3 \cdot (b^2)^3 = a^{1 \cdot 3} \cdot b^{2 \cdot 3} = a^3 b^6$ Ⓑ $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15} t^{15} = 32,768 t^{15}$
 Ⓒ $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3 \cdot 3} = -8w^9$ Ⓓ $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \cdot (z)^4} = \frac{1}{2,401z^4}$
 Ⓔ $(e^{-2}f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2 \cdot 7} \cdot f^{2 \cdot 7} = e^{-14} f^{14} = \frac{f^{14}}{e^{14}}$

> **TRY IT** #7 Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

Ⓐ $(g^2h^3)^5$ Ⓑ $(5t)^3$ Ⓒ $(-3y^5)^3$ Ⓓ $\frac{1}{(a^6b^7)^3}$ Ⓔ $(r^3s^{-2})^4$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule*, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$(e^{-2}f^2)^7 = \frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}
 (e^{-2} f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\
 &= \frac{(f^2)^7}{(e^2)^7} \\
 &= \frac{f^{2 \cdot 7}}{e^{2 \cdot 7}} \\
 &= \frac{f^{14}}{e^{14}}
 \end{aligned}$$

The Power of a Quotient Rule of Exponents

For any real numbers a and b and any integer n , the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

EXAMPLE 8

Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a $\left(\frac{4}{z^{11}}\right)^3$ b $\left(\frac{p}{q^3}\right)^6$ c $\left(\frac{-1}{t^2}\right)^{27}$ d $(j^3 k^{-2})^4$ e $(m^{-2} n^{-2})^3$

✓ Solution

a $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{(z^{11})^3} = \frac{64}{z^{11 \cdot 3}} = \frac{64}{z^{33}}$ b $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{(q^3)^6} = \frac{p^{1 \cdot 6}}{q^{3 \cdot 6}} = \frac{p^6}{q^{18}}$

c $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{(t^2)^{27}} = \frac{-1}{t^{2 \cdot 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$ d $(j^3 k^{-2})^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{(j^3)^4}{(k^2)^4} = \frac{j^{3 \cdot 4}}{k^{2 \cdot 4}} = \frac{j^{12}}{k^8}$

e $(m^{-2} n^{-2})^3 = \left(\frac{1}{m^2 n^2}\right)^3 = \frac{(1)^3}{(m^2 n^2)^3} = \frac{1}{(m^2)^3 (n^2)^3} = \frac{1}{m^{2 \cdot 3} \cdot n^{2 \cdot 3}} = \frac{1}{m^6 n^6}$

> **TRY IT** #8 Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a $\left(\frac{b^5}{c}\right)^3$ b $\left(\frac{5}{u^8}\right)^4$ c $\left(\frac{-1}{w^3}\right)^{35}$ d $(p^{-4} q^3)^8$ e $(c^{-5} d^{-3})^4$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combining terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

EXAMPLE 9

Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a $(6m^2 n^{-1})^3$ b $17^5 \cdot 17^{-4} \cdot 17^{-3}$ c $\left(\frac{u^{-1} v}{v^{-1}}\right)^2$ d $(-2a^3 b^{-1})(5a^{-2} b^2)$

e $(x^2 \sqrt{2})^4 (x^2 \sqrt{2})^{-4}$ f $\frac{(3w^2)^5}{(6w^{-2})^2}$

✓ **Solution**

(a)

$$\begin{aligned}(6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 \\ &= 6^3m^{2 \cdot 3}n^{-1 \cdot 3} \\ &= 216m^6n^{-3} \\ &= \frac{216m^6}{n^3}\end{aligned}$$

The power of a product rule

The power rule

Simplify.

The negative exponent rule

(b)

$$\begin{aligned}17^5 \cdot 17^{-4} \cdot 17^{-3} &= 17^{5-4-3} \\ &= 17^{-2} \\ &= \frac{1}{17^2} \text{ or } \frac{1}{289}\end{aligned}$$

The product rule

Simplify.

The negative exponent rule

(c)

$$\begin{aligned}\left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} \\ &= \frac{u^{-2}v^2}{v^{-2}} \\ &= u^{-2}v^{2-(-2)} \\ &= u^{-2}v^4 \\ &= \frac{v^4}{u^2}\end{aligned}$$

The power of a quotient rule

The power of a product rule

The quotient rule

Simplify.

The negative exponent rule

(d)

$$\begin{aligned}(-2a^3b^{-1})(5a^{-2}b^2) &= -2 \cdot 5 \cdot a^3 \cdot a^{-2} \cdot b^{-1} \cdot b^2 \\ &= -10 \cdot a^{3-2} \cdot b^{-1+2} \\ &= -10ab\end{aligned}$$

Commutative and associative laws of multiplication

The product rule

Simplify.

(e)

$$\begin{aligned}(x^2\sqrt{2})^4(x^2\sqrt{2})^{-4} &= (x^2\sqrt{2})^{4-4} \\ &= (x^2\sqrt{2})^0 \\ &= 1\end{aligned}$$

The product rule

Simplify.

The zero exponent rule

(f)

$$\begin{aligned}\frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5 \cdot (w^2)^5}{(6)^2 \cdot (w^{-2})^2} \\ &= \frac{3^5 w^{2 \cdot 5}}{6^2 w^{-2 \cdot 2}} \\ &= \frac{243w^{10}}{36w^{-4}} \\ &= \frac{27w^{10-(-4)}}{4} \\ &= \frac{27w^{14}}{4}\end{aligned}$$

The power of a product rule

The power rule

Simplify.

The quotient rule and reduce fraction

Simplify.



TRY IT

#9

Simplify each expression and write the answer with positive exponents only.

(a) $(2uw^{-2})^{-3}$ (b) $x^8 \cdot x^{-12} \cdot x$ (c) $\left(\frac{e^2 f^{-3}}{f^{-1}}\right)^2$ (d) $(9r^{-5}s^3)(3r^6s^{-4})$

(e) $\left(\frac{4}{9}tw^{-2}\right)^{-3}\left(\frac{4}{9}tw^{-2}\right)^3$ (f) $\frac{(2h^2k)^4}{(7h^{-1}k^2)^2}$

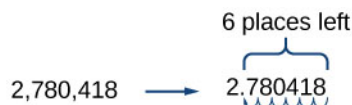
Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an

electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n . If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.



We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$2.780418 \times 10^6$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.



Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

Scientific Notation

A number is written in **scientific notation** if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

EXAMPLE 10

Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

- (a) Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000 m
- (b) Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000 m
- (c) Number of stars in Andromeda Galaxy: 1,000,000,000,000
- (d) Diameter of electron: 0.00000000000094 m
- (e) Probability of being struck by lightning in any single year: 0.00000143

✓ Solution

- (a) 24,000,000,000,000,000,000 m
24,000,000,000,000,000,000 m

←22 places

$$2.4 \times 10^{22} \text{ m}$$

- (b)

1,300,000,000,000,000,000 m

1,300,000,000,000,000,000 m

←21 places

 1.3×10^{21} m

Ⓒ

1,000,000,000,000

1,000,000,000,000

←12 places

 1×10^{12}

Ⓓ

0.00000000000094 m

0.00000000000094 m

→13 places

 9.4×10^{-13} m

Ⓔ

0.00000143

0.00000143

→6 places

 1.43×10^{-6} Ⓒ **Analysis**

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

> **TRY IT** #10 Write each number in scientific notation.

- Ⓐ U.S. national debt per taxpayer (April 2014): \$152,000
- Ⓑ World population (April 2014): 7,158,000,000
- Ⓒ World gross national income (April 2014): \$85,500,000,000,000
- Ⓓ Time for light to travel 1 m: 0.0000000334 s
- Ⓔ Probability of winning lottery (match 6 of 49 possible numbers): 0.000000715

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

EXAMPLE 11

Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

- Ⓐ 3.547×10^{14} Ⓑ -2×10^6 Ⓒ 7.91×10^{-7} Ⓓ -8.05×10^{-12}

✓ **Solution**

Ⓐ	Ⓑ	Ⓒ	Ⓓ
3.547×10^{14}	-2×10^6	7.91×10^{-7}	-8.05×10^{-12}
3.54700000000000	-2.000000	0000007.91	-000000000008.05
→14 places	→6 places	→7 places	→12 places
354,700,000,000,000	-2,000,000	0.000000791	-0.00000000000805

> **TRY IT** #11 Convert each number in scientific notation to standard notation.

- (a) 7.03×10^5 (b) -8.16×10^{11} (c) -3.9×10^{-13} (d) 8×10^{-6}

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \cdot (1.32 \times 10^{21}) \cdot (1.22 \times 10^4) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^4) \cdot (5 \times 10^6) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10 . That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

EXAMPLE 12

Using Scientific Notation

Perform the operations and write the answer in scientific notation.

- (a) $(8.14 \times 10^{-7})(6.5 \times 10^{10})$ (b) $(4 \times 10^5) \div (-1.52 \times 10^9)$ (c) $(2.7 \times 10^5)(6.04 \times 10^{13})$
 (d) $(1.2 \times 10^8) \div (9.6 \times 10^5)$ (e) $(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5)$

Solution

(a)

$$\begin{aligned} (8.14 \times 10^{-7})(6.5 \times 10^{10}) &= (8.14 \times 6.5)(10^{-7} \times 10^{10}) \\ &= (52.91)(10^3) \\ &= 5.291 \times 10^4 \end{aligned}$$

Commutative and associative properties of multiplication
Product rule of exponents
Scientific notation

(b)

$$\begin{aligned} (4 \times 10^5) \div (-1.52 \times 10^9) &= \left(\frac{4}{-1.52}\right) \left(\frac{10^5}{10^9}\right) \\ &\approx (-2.63)(10^{-4}) \\ &= -2.63 \times 10^{-4} \end{aligned}$$

Commutative and associative properties of multiplication
Quotient rule of exponents
Scientific notation

(c)

$$\begin{aligned} (2.7 \times 10^5)(6.04 \times 10^{13}) &= (2.7 \times 6.04)(10^5 \times 10^{13}) \\ &= (16.308)(10^{18}) \\ &= 1.6308 \times 10^{19} \end{aligned}$$

Commutative and associative properties of multiplication
Product rule of exponents
Scientific notation

(d)

$$\begin{aligned} (1.2 \times 10^8) \div (9.6 \times 10^5) &= \left(\frac{1.2}{9.6}\right) \left(\frac{10^8}{10^5}\right) \\ &= (0.125)(10^3) \\ &= 1.25 \times 10^2 \end{aligned}$$

Commutative and associative properties of multiplication
Quotient rule of exponents
Scientific notation

(e)

$$\begin{aligned} (3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5) &= [3.33 \times (-1.05) \times 5.62](10^4 \times 10^7 \times 10^5) \\ &\approx (-19.65)(10^{16}) \\ &= -1.965 \times 10^{17} \end{aligned}$$

> **TRY IT** #12 Perform the operations and write the answer in scientific notation.

- a $(-7.5 \times 10^8)(1.13 \times 10^{-2})$ b $(1.24 \times 10^{11}) \div (1.55 \times 10^{18})$
 c $(3.72 \times 10^9)(8 \times 10^3)$ d $(9.933 \times 10^{23}) \div (-2.31 \times 10^{17})$
 e $(-6.04 \times 10^9)(7.3 \times 10^2)(-2.81 \times 10^2)$

EXAMPLE 13**Applying Scientific Notation to Solve Problems**

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about \$17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

✓ Solution

The population was $308,000,000 = 3.08 \times 10^8$.

The national debt was $\$17,547,000,000,000 \approx \1.75×10^{13} .

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned} (1.75 \times 10^{13}) \div (3.08 \times 10^8) &= \left(\frac{1.75}{3.08}\right) \cdot \left(\frac{10^{13}}{10^8}\right) \\ &\approx 0.57 \times 10^5 \\ &= 5.7 \times 10^4 \end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$, or \$57,000.

> TRY IT #13 An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

▶ MEDIA

Access these online resources for additional instruction and practice with exponents and scientific notation.

- [Exponential Notation \(http://openstax.org/l/exponnot\)](http://openstax.org/l/exponnot)
- [Properties of Exponents \(http://openstax.org/l/exponprops\)](http://openstax.org/l/exponprops)
- [Zero Exponent \(http://openstax.org/l/zeroexponent\)](http://openstax.org/l/zeroexponent)
- [Simplify Exponent Expressions \(http://openstax.org/l/exponexpres\)](http://openstax.org/l/exponexpres)
- [Quotient Rule for Exponents \(http://openstax.org/l/quotofexpon\)](http://openstax.org/l/quotofexpon)
- [Scientific Notation \(http://openstax.org/l/scientificnota\)](http://openstax.org/l/scientificnota)
- [Converting to Decimal Notation \(http://openstax.org/l/decimalnota\)](http://openstax.org/l/decimalnota)

1.2 SECTION EXERCISES

Verbal

1. Is 2^3 the same as 3^2 ? Explain.
2. When can you add two exponents?
3. What is the purpose of scientific notation?
4. Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

5. 9^2

6. 15^{-2}

7. $3^2 \times 3^3$

8. $4^4 \div 4$

9. $(2^2)^{-2}$

10. $(5 - 8)^0$

11. $11^3 \div 11^4$

12. $6^5 \times 6^{-7}$

13. $(8^0)^2$

14. $5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

15. $4^2 \times 4^3 \div 4^{-4}$

16. $\frac{6^{12}}{6^9}$

17. $(12^3 \times 12)^{10}$

18. $10^6 \div (10^{10})^{-2}$

19. $7^{-6} \times 7^{-3}$

20. $(3^3 \div 3^4)^5$

For the following exercises, express the decimal in scientific notation.

21. 0.0000314

22. 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

23. 1.6×10^{10}

24. 9.8×10^{-9}

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

25. $\frac{a^3 a^2}{a}$

26. $\frac{mn^2}{m^{-2}}$

27. $(b^3 c^4)^2$

28. $\left(\frac{x^{-3}}{y^2}\right)^{-5}$

29. $ab^2 \div d^{-3}$

30. $(w^0 x^5)^{-1}$

31. $\frac{m^4}{n^0}$

32. $y^{-4} (y^2)^2$

33. $\frac{p^{-4} q^2}{p^2 q^{-3}}$

34. $(l \times w)^2$

35. $(y^7)^3 \div x^{14}$

36. $\left(\frac{a}{2^3}\right)^2$

37. $(25m) \div (\frac{5}{0}m)$

38. $\frac{(16\sqrt{x})^2}{y^{-1}}$

39. $\frac{2^3}{(3a)^{-2}}$

40. $(ma^6)^2 \frac{1}{m^3 a^2}$

41. $(b^{-3}c)^3$

42. $(x^2 y^{13} \div y^0)^2$

43. $(9z^3)^{-2}y$

Real-World Applications

44. To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

45. A dime is the thinnest coin in U.S. currency. A dime's thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

46. The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

47. A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

48. The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

49. One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

50. The value of the services sector of the U.S. economy in the first quarter of 2012 was \$10,633.6 billion. Rewrite this amount in scientific notation.

Technology

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

51. $\left(\frac{12^3 m^{33}}{4^{-3}}\right)^2$

52. $17^3 \div 15^2 x^3$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

53. $\left(\frac{3^2}{a^3}\right)^{-2} \left(\frac{a^4}{2^2}\right)^2$

54. $(6^2 - 24)^2 \div \left(\frac{x}{y}\right)^{-5}$

55. $\frac{m^2 n^3}{a^2 c^{-3}} \cdot \frac{a^{-7} n^{-2}}{m^2 c^4}$

56. $\left(\frac{x^6 y^3}{x^3 y^{-3}} \cdot \frac{y^{-7}}{x^{-3}}\right)^{10}$

57. $\left(\frac{(ab^2c)^{-3}}{b^{-3}}\right)^2$

58. Avogadro's constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23} . Write Avogadro's constant in standard notation.

3.2 Modeling with exponential functions

Motivating Questions

- What can we say about the behavior of an exponential function as the input gets larger and larger?
- How do vertical stretches and shifts of an exponential function affect its behavior?
- Why is the temperature of a cooling or warming object modeled by a function of the form $F(t) = ab^t + c$?

If a quantity changes so that its growth or decay occurs at a constant percentage rate with respect to time, the function is exponential. This is because if the growth or decay rate is r , the total amount of the quantity at time t is given by $A(t) = a(1 + r)^t$, where a is the amount present at time $t = 0$. Many different natural quantities change according to exponential models: money growth through compounding interest, the growth of a population of cells, and the decay of radioactive elements.

A related situation arises when an object's temperature changes in response to its surroundings. For instance, if we have a cup of coffee at an initial temperature of 186° Fahrenheit and the cup is placed in a room where the surrounding temperature is 71° , our intuition and experience tell us that over time the coffee will cool and eventually tend to the 71° temperature of the surroundings. From an experiment¹ with an actual temperature probe, we have the data in Table 3.2.1 that is plotted in Figure 3.2.2.

t	0	1	2	3	8	13
$F(t)$	186	179	175	171	156	144
	18	23	28	33	38	43
	135	127	120	116	111	107

Table 3.2.1: Data for cooling coffee, measured in degrees Fahrenheit at time t in minutes.

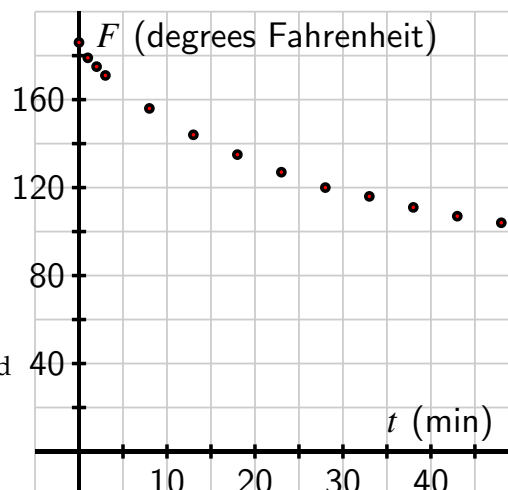


Figure 3.2.2: A plot of the data in Table 3.2.1.

In one sense, the data looks exponential: the points appear to lie on a curve that is always decreasing and decreasing at an increasing rate. However, we know that the function can't

¹See <http://gvsu.edu/s/OSB> for this data.

have the form $f(t) = ab^t$ because such a function's range is the set of all positive real numbers, and it's impossible for the coffee's temperature to fall below room temperature (71°). It is natural to wonder if a function of the form $g(t) = ab^t + c$ will work. Thus, in order to find a function that fits the data in a situation such as Figure 3.2.2, we begin by investigating and understanding the roles of a , b , and c in the behavior of $g(t) = ab^t + c$.

Preview Activity 3.2.1. In *Desmos*, define $g(t) = ab^t + c$ and accept the prompt for sliders for both a and b . Edit the sliders so that a has values from $a = 5$ to $a = 50$, b has values from $b = 0.7$ to $b = 1.3$, and c has values from $c = -5$ to $c = 5$ (also with a step-size of 0.01). In addition, in *Desmos* let $P = (0, g(0))$ and check the box to show the label. Finally, zoom out so that the window shows an interval of t -values from $-30 \leq t \leq 30$.

- Set $b = 1.1$ and explore the effects of changing the values of a and c . Write several sentences to summarize your observations.
- Follow the directions for (a) again, this time with $b = 0.9$
- Set $a = 5$ and $c = 4$. Explore the effects of changing the value of b ; be sure to include values of b both less than and greater than 1. Write several sentences to summarize your observations.
- When $0 < b < 1$, what happens to the graph of g when we consider positive t -values that get larger and larger?

3.2.1 Long-term behavior of exponential functions

We have already established that any exponential function of the form $f(t) = ab^t$ where a and b are positive real numbers with $b \neq 1$ is always concave up and is either always increasing or always decreasing. We next introduce precise language to describe the behavior of an exponential function's value as t gets bigger and bigger. To start, let's consider the two basic exponential functions $p(t) = 2^t$ and $q(t) = (\frac{1}{2})^t$ and their respective values at $t = 10$, $t = 20$, and $t = 30$, as displayed in Table 3.2.3 and Table 3.2.4.

t	$p(t)$
10	$2^{10} = 1026$
20	$2^{20} = 1048576$
30	$2^{30} = 1073741824$

Table 3.2.3: Data for $p(t) = 2^t$.

t	$q(t)$
10	$(\frac{1}{2})^{10} = \frac{1}{1026} \approx 0.00097656$
20	$(\frac{1}{2})^{20} = \frac{1}{1048576} \approx 0.0000095367$
30	$(\frac{1}{2})^{30} = \frac{1}{1073741824} \approx 0.0000000093192$

Table 3.2.4: Data for $q(t) = (\frac{1}{2})^t$.

For the increasing function $p(t) = 2^t$, we see that the output of the function gets very large very quickly. In addition, there is no upper bound to how large the function can be. Indeed, we can make the value of $p(t)$ as large as we'd like by taking t sufficiently big. We thus say that as t increases, $p(t)$ **increases without bound**.

For the decreasing function $q(t) = (\frac{1}{2})^t$, we see that the output $q(t)$ is always positive but getting closer and closer to 0. Indeed, because we can make 2^t as large as we like, it follows

that we can make its reciprocal $\frac{1}{2^t} = \left(\frac{1}{2}\right)^t$ as small as we'd like. We thus say that as t increases, $q(t)$ **approaches 0**.

To represent these two common phenomena with exponential functions—the value increasing without bound or the value approaching 0—we will use shorthand notation. First, it is natural to write “ $q(t) \rightarrow 0$ ” as t increases without bound. Moreover, since we have the notion of the infinite to represent quantities without bound, we use the symbol for infinity and arrow notation (∞) and write “ $p(t) \rightarrow \infty$ ” as t increases without bound in order to indicate that $p(t)$ increases without bound.

In Preview Activity 3.2.1, we saw how the value of b affects the steepness of the graph of $f(t) = ab^t$, as well as how all graphs with $b > 1$ have the similar increasing behavior, and all graphs with $0 < b < 1$ have similar decreasing behavior. For instance, by taking t sufficiently large, we can make $(1.01)^t$ as large as we want; it just takes much larger t to make $(1.01)^t$ big in comparison to 2^t . In the same way, we can make $(0.99)^t$ as close to 0 as we wish by taking t sufficiently big, even though it takes longer for $(0.99)^t$ to get close to 0 in comparison to $\left(\frac{1}{2}\right)^t$. For an arbitrary choice of b , we can say the following.

Long-term behavior of exponential functions.

Let $f(t) = b^t$ with $b > 0$ and $b \neq 1$.

- If $0 < b < 1$, then $b^t \rightarrow 0$ as $t \rightarrow \infty$. We read this notation as “ b^t tends to 0 as t increases without bound.”
- If $b > 1$, then $b^t \rightarrow \infty$ as $t \rightarrow \infty$. We read this notation as “ b^t increases without bound as t increases without bound.”

In addition, we make a key observation about the use of exponents. For the function $q(t) = \left(\frac{1}{2}\right)^t$, there are three equivalent ways we may write the function:

$$\left(\frac{1}{2}\right)^t = \frac{1}{2^t} = 2^{-t}.$$

In our work with transformations involving horizontal scaling in Exercise 2.4.5.9, we saw that the graph of $y = h(-t)$ is the reflection of the graph of $y = h(t)$ across the y -axis. Therefore, we can say that the graphs of $p(t) = 2^t$ and $q(t) = \left(\frac{1}{2}\right)^t = 2^{-t}$ are reflections of one another in the y -axis since $p(-t) = 2^{-t} = q(t)$. We see this fact verified in Figure 3.2.5. Similar observations hold for the relationship between the graphs of b^t and $\frac{1}{b^t} = b^{-t}$ for any positive $b \neq 1$.

3.2.2 The role of c in $g(t) = ab^t + c$

The function $g(t) = ab^t + c$ is a vertical translation of the function $f(t) = ab^t$. We now have extensive understanding of the behavior of $f(t)$ and how that behavior depends on a and b . Since a vertical translation by c does not change the shape of any graph, we expect that g will exhibit very similar behavior to f . Indeed, we can compare the two functions' graphs as shown in Figure 3.2.6 and Figure 3.2.7 and then make the following general observations.

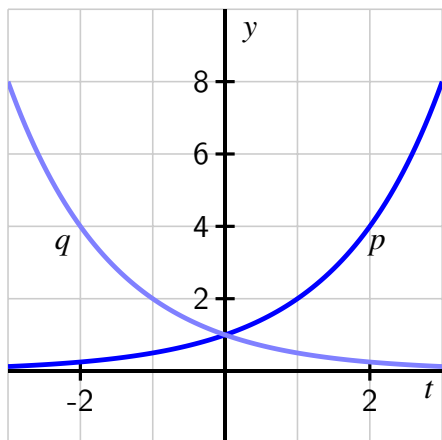


Figure 3.2.5: Plots of $p(t) = 2^t$ and $q(t) = 2^{-t}$.

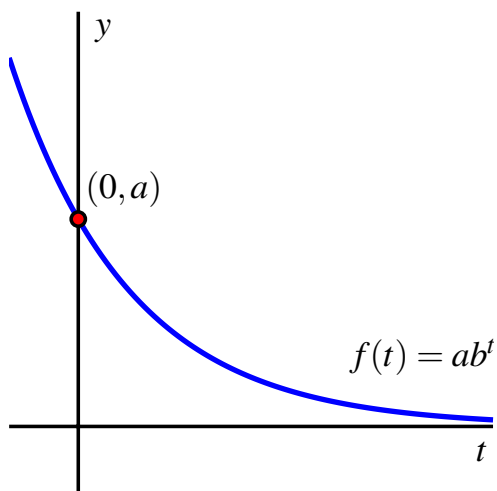


Figure 3.2.6: Plot of $f(t) = ab^t$.

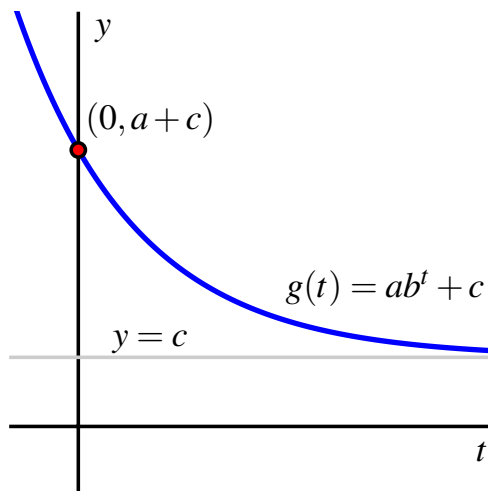


Figure 3.2.7: Plot of $g(t) = ab^t + c$.

Behavior of vertically shifted exponential functions.

Let $g(t) = ab^t + c$ with $a > 0$, $b > 0$ and $b \neq 1$, and c any real number.

- If $0 < b < 1$, then $g(t) = ab^t + c \rightarrow c$ as $t \rightarrow \infty$. The function g is always decreasing, always concave up, and has y -intercept $(0, a + c)$. The range of the function is all real numbers greater than c .
- If $b > 1$, then $g(t) = ab^t + c \rightarrow \infty$ as $t \rightarrow \infty$. The function g is always increasing, always concave up, and has y -intercept $(0, a + c)$. The range of the function is all real numbers greater than c .

It is also possible to have $a < 0$. In this situation, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x -axis and a vertical stretch by $|a|$, the function g is always concave down. If $0 < b < 1$ so that f is always decreasing, then g is always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Moreover, instead of the range of the function g having a lower bound as when $a > 0$, in this setting the range of g has an upper bound. These ideas are explored further in Activity 3.2.2.

It's an important skill to be able to look at an exponential function of the form $g(t) = ab^t + c$ and form an accurate mental picture of the graph's main features in light of the values of a , b , and c .

Activity 3.2.2. For each of the following functions, *without* using graphing technology, determine whether the function is

- i. always increasing or always decreasing;
- ii. always concave up or always concave down; and
- iii. increasing without bound, decreasing without bound, or increasing/decreasing toward a finite value.

In addition, state the y -intercept and the range of the function. For each function, write a sentence that explains your thinking and sketch a rough graph of how the function appears.

a. $p(t) = 4372(1.000235)^t + 92856$

d. $s(t) = -17398(0.85234)^t + 19411$

b. $q(t) = 27931(0.97231)^t + 549786$

e. $u(t) = -7522(1.03817)^t$

c. $r(t) = -17398(0.85234)^t$

f. $v(t) = -7522(1.03817)^t + 6731$

3.2.3 Modeling temperature data

Newton's Law of Cooling states that the rate that an object warms or cools occurs in direct proportion to the difference between its own temperature and the temperature of its surroundings. If we return to the coffee temperature data in Table 3.2.1 and recall that the room temperature in that experiment was 71° , we can see how to use a transformed exponential function to model the data. In Table 3.2.8, we add a row of information to the table where we compute $F(t) - 71$ to subtract the room temperature from each reading.

t	0	1	2	3	8	13	18	23	28	33	38	43	48
$F(t)$	186	179	175	171	156	144	135	127	120	116	111	107	104
$f(t) = F(t) - 71$	115	108	104	100	85	73	64	56	49	45	40	36	33

Table 3.2.8: Data for cooling coffee, measured in degrees Fahrenheit at time t in minutes, plus shifted to account for room temperature.

The data in the bottom row of Table 3.2.8 appears exponential, and if we test the data by

computing the quotients of output values that correspond to equally-spaced input, we see a nearly constant ratio. In particular,

$$\frac{73}{85} \approx 0.86, \quad \frac{64}{73} \approx 0.88, \quad \frac{56}{64} \approx 0.88, \quad \frac{49}{56} \approx 0.88, \quad \frac{45}{49} \approx 0.92, \quad \text{and} \quad \frac{40}{45} \approx 0.89.$$

Of course, there is some measurement error in the data (plus it is only recorded to accuracy of whole degrees), so these computations provide convincing evidence that the underlying function is exponential. In addition, we expect that if the data continued in the bottom row of Table 3.2.8, the values would approach 0 because $F(t)$ will approach 71.

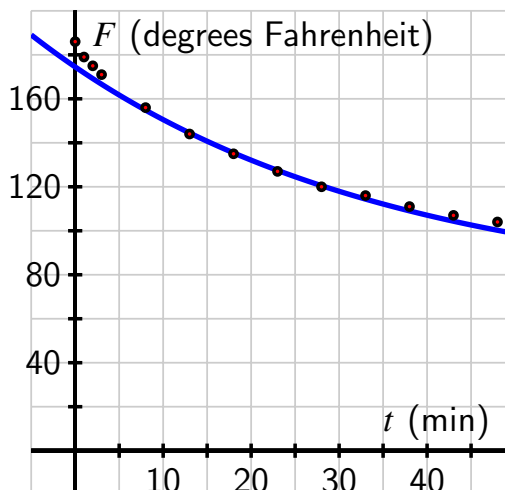
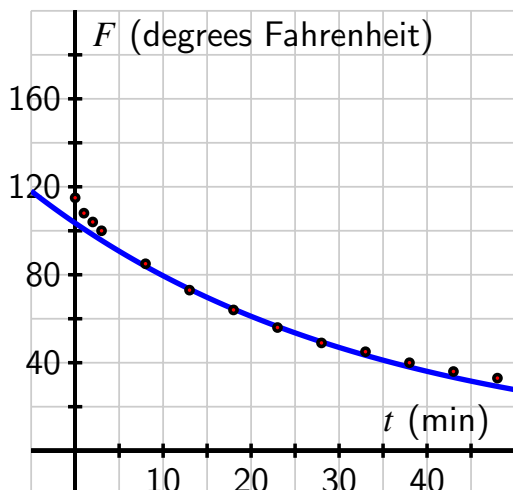


Figure 3.2.9: Plot of $f(t) = 103.503(0.974)^t$. **Figure 3.2.10:** Plot of $F(t) = 103.503(0.974)^t + 71$.

If we choose two of the data points, say $(18, 64)$ and $(23, 56)$, and assume that $f(t) = ab^t$, we can determine the values of a and b . Doing so, it turns out that $a \approx 103.503$ and $b \approx 0.974$, so $f(t) = 103.503(0.974)^t$. Since $f(t) = F(t) - 71$, we see that $F(t) = f(t) + 71$, so $F(t) = 103.503(0.974)^t + 71$. Plotting f against the shifted data and F along with the original data in Figure 3.2.9 and Figure 3.2.10, we see that the curves go exactly through the points where $t = 18$ and $t = 23$ as expected, but also that the function provides a reasonable model for the observed behavior at any time t . If our data was even more accurate, we would expect that the curve's fit would be even better.

Our preceding work with the coffee data can be done similarly with data for any cooling or warming object whose temperature initially differs from its surroundings. Indeed, it is possible to show that Newton's Law of Cooling implies that the object's temperature is given by a function of the form $F(t) = ab^t + c$.

Activity 3.2.3. A can of soda (at room temperature) is placed in a refrigerator at time $t = 0$ (in minutes) and its temperature, $F(t)$, in degrees Fahrenheit, is computed at regular intervals. Based on the data, a model is formulated for the object's temperature, given by

$$F(t) = 42 + 30(0.95)^t.$$

- Consider the simpler (parent) function $p(t) = (0.95)^t$. How do you expect the graph of this function to appear? How will it behave as time increases? Without using graphing technology, sketch a rough graph of p and write a sentence of explanation.
- For the slightly more complicated function $r(t) = 30(0.95)^t$, how do you expect this function to look in comparison to p ? What is the long-range behavior of this function as t increases? Without using graphing technology, sketch a rough graph of r and write a sentence of explanation.
- Finally, how do you expect the graph of $F(t) = 42 + 30(0.95)^t$ to appear? Why? First sketch a rough graph without graphing technology, and then use technology to check your thinking and report an accurate, labeled graph on the axes provided in Figure 3.2.11.

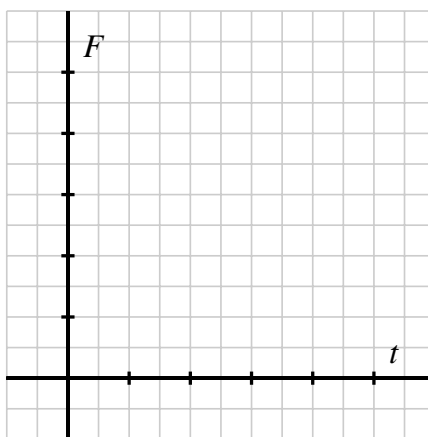


Figure 3.2.11: Axes for plotting F .

- What is the temperature of the refrigerator? What is the room temperature of the surroundings outside the refrigerator? Why?
- Determine the average rate of change of F on the intervals $[10, 20]$, $[20, 30]$, and $[30, 40]$. Write at least two careful sentences that explain the meaning of the values you found, including units, and discuss any overall trend in how the average rate of change is changing.

Activity 3.2.4. A potato initially at room temperature (68°) is placed in an oven (at 350°) at time $t = 0$. It is known that the potato's temperature at time t is given by the function $F(t) = a - b(0.98)^t$ for some positive constants a and b , where F is measured

in degrees Fahrenheit and t is time in minutes.

- What is the numerical value of $F(0)$? What does this tell you about the value of $a - b$?
- Based on the context of the problem, what should be the long-range behavior of the function $F(t)$? Use this fact along with the behavior of $(0.98)^t$ to determine the value of a . Write a sentence to explain your thinking.
- What is the value of b ? Why?
- Check your work above by plotting the function F using graphing technology in an appropriate window. Record your results on the axes provided in Figure 3.2.12, labeling the scale on the axes. Then, use the graph to estimate the time at which the potato's temperature reaches 325 degrees.

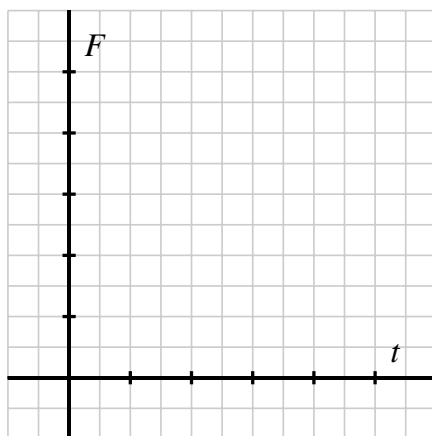


Figure 3.2.12: Axes for plotting F .

- How can we view the function $F(t) = a - b(0.98)^t$ as a transformation of the parent function $f(t) = (0.98)^t$? Explain.

3.2.4 Summary

- For an exponential function of the form $f(t) = b^t$, the function either approaches zero or grows without bound as the input gets larger and larger. In particular, if $0 < b < 1$, then $f(t) = b^t \rightarrow 0$ as $t \rightarrow \infty$, while if $b > 1$, then $f(t) = b^t \rightarrow \infty$ as $t \rightarrow \infty$. Scaling f by a positive value a (that is, the transformed function ab^t) does not affect the long-range behavior: whether the function tends to 0 or increases without bound depends solely on whether b is less than or greater than 1.
- The function $f(t) = b^t$ passes through $(0, 1)$, is always concave up, is either always

increasing or always decreasing, and its range is the set of all positive real numbers. Among these properties, a vertical stretch by a positive value a only affects the y -intercept, which is instead $(0, a)$. If we include a vertical shift and write $g(t) = ab^t + c$, the biggest change is that the range of g is the set of all real numbers greater than c . In addition, the y -intercept of g is $(0, a + c)$.

In the situation where $a < 0$, several other changes are induced. Here, because $g(t) = ab^t$ is both a reflection of $f(t) = b^t$ across the x -axis and a vertical stretch by $|a|$, the function g is now always concave down. If $0 < b < 1$ so that f is always decreasing, then g (the reflected function) is now always increasing; if instead $b > 1$ so f is increasing, then g is decreasing. Finally, if $a < 0$, then the range of $g(t) = ab^t + c$ is the set of all real numbers c .

- An exponential function can be thought of as a function that changes at a rate proportional to itself, like how money grows with compound interest or the amount of a radioactive quantity decays. Newton's Law of Cooling says that the rate of change of an object's temperature is proportional to the *difference* between its own temperature and the temperature of its surroundings. This leads to the function that measures the difference between the object's temperature and room temperature being exponential, and hence the object's temperature itself is a vertically-shifted exponential function of the form $F(t) = ab^t + c$.

3.2.5 Exercises

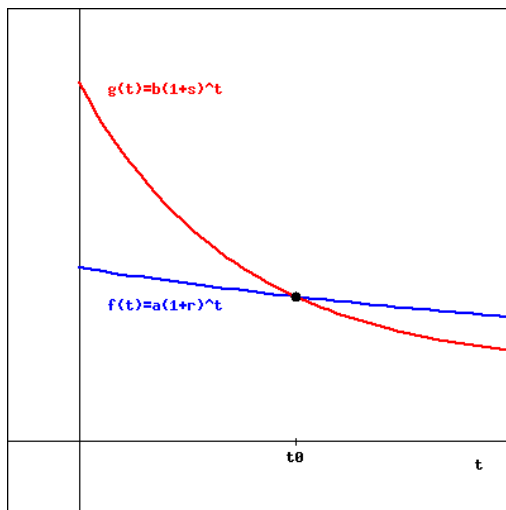
1. If $b > 1$, what is the horizontal asymptote of $y = ab^t$ as $t \rightarrow -\infty$?
2. Find the long run behavior of each of the following functions.
 - (a) As $x \rightarrow \infty$, $18(0.8)^x \rightarrow$ _____
 - (b) As $t \rightarrow -\infty$, $9(2.2)^t \rightarrow$ _____
 - (c) As $t \rightarrow \infty$, $0.6(2 + (0.1)^t) \rightarrow$ _____





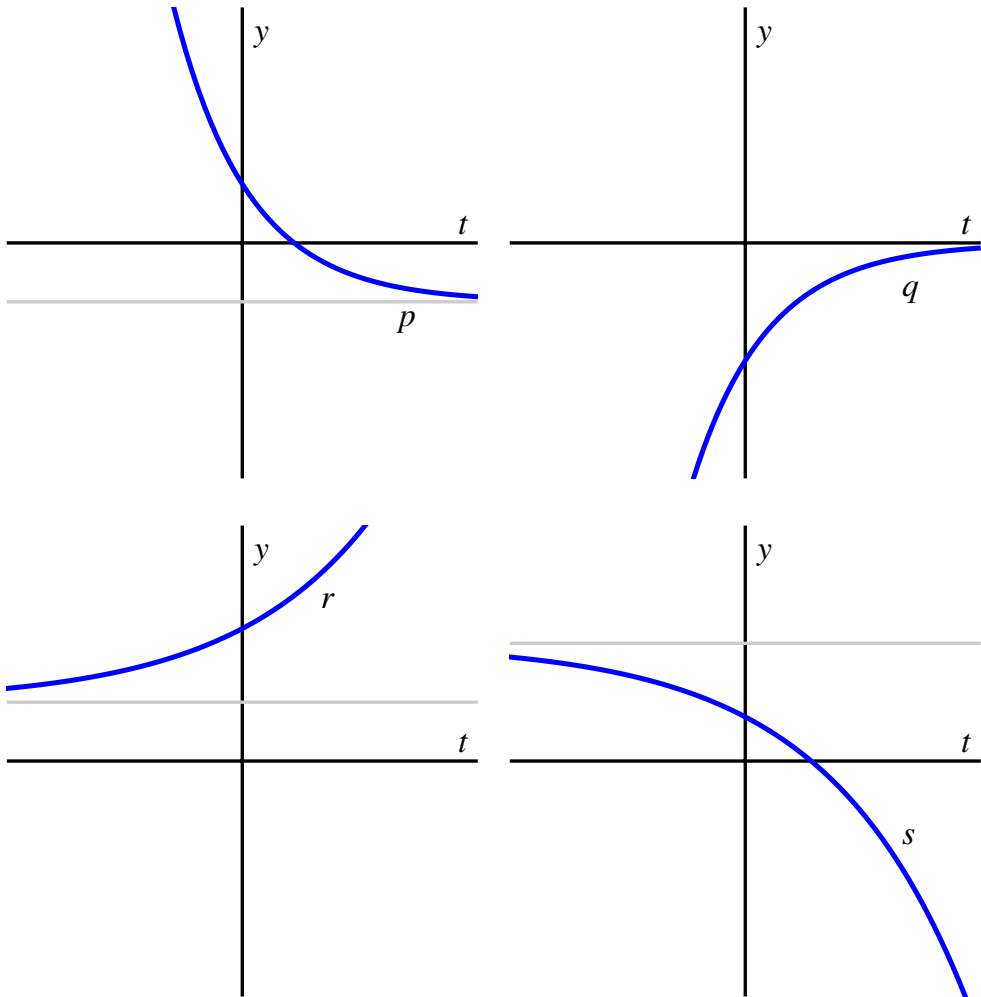
3. Suppose t_0 is the t -coordinate of the point of intersection of the graphs below. Complete the statement below in order to correctly describe what happens to t_0 if the value of r (in the blue graph of $f(t) = a(1+r)^t$ below) is increased, and all other quantities remain the same.

As r increases, does the value of t_0 increase, decrease, or remain the same?



4. A can of soda has been in a refrigerator for several days; the refrigerator has temperature 41° Fahrenheit. Upon removal, the soda is placed on a kitchen table in a room with surrounding temperature 72° . Let $F(t)$ represent the soda's temperature in degrees Fahrenheit at time t in minutes, where $t = 0$ corresponds to the time the can is removed from the refrigerator. We know from Newton's Law of Cooling that F has form $F(t) = ab^t + c$ for some constants a , b , and c , where $0 < b < 1$.
- What is the numerical value of the soda's initial temperature? What is the value of $F(0)$ in terms of a , b , and c ? What do these two observations tell us?
 - What is the numerical value of the soda's long-term temperature? What is the long-term value of $F(t)$ in terms of a , b , and c ? What do these two observations tell us?
 - Using your work in (a) and (b), determine the numerical values of a and c .
 - Suppose it can be determined that $b = 0.931$. What is the soda's temperature after 10 minutes?

5. Consider the graphs of the following four functions p , q , r , and s . Each is a shifted exponential function of the form $ab^t + c$.



For each function p , q , r , and s , determine

- whether $a > 0$ or $a < 0$;
 - whether $0 < b < 1$ or $b > 1$;
 - whether $c > 0$, $c = 0$, or $c < 0$; and
 - the range of the function in terms of c .
6. A cup of coffee has its temperature, $C(t)$, measured in degrees Celsius. When poured outdoors on a cold morning, its temperature is $C(0) = 95$. Ten minutes later, $C(10) = 80$. If the surrounding temperature outside is 0° Celsius, find a formula for a function $C(t)$ that models the coffee's temperature at time t .

In addition, recall that we can convert between Celsius and Fahrenheit according to the

Chapter 3 Exponential and Logarithmic Functions

equations $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$. Use this information to also find a formula for $F(t)$, the coffee's Fahrenheit temperature at time t . What is similar and what is different regarding the functions $C(t)$ and $F(t)$?

4.3 linear and rational equations (OpenStax College Algebra with Corequisite Support)

4.3.1 solving equations

4.3.2 writing an equation of a line

4.3.3 parallel and perpendicular lines

Objective 1: Simplify expressions using order of operations (IA 1.1.3)**HOW TO**

Use the order of operations

Step 1. Parentheses and Other Grouping Symbols

- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

Step 2. Exponents

- Simplify all expressions with exponents.

Step 3. Multiplication and Division

- Perform all multiplication and division in order from left to right. These operations have equal priority.

Step 4. Addition and Subtraction

- Perform all addition and subtraction in order from left to right. These operations have equal priority.

EXAMPLE 1

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

Solution

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 0$
Is there any addition or subtraction? Yes.	
Add.	$13 + 0$
Add.	13

Practice Makes Perfect

1. $3(1 + 9 \cdot 6) - 4^2$

2. $2^3 - 12 \div (9 - 5)$

3. $33 \div 3 + 4(7 - 2)$

4. $10 + 3[6 - 2(4 - 2)] - 2^4$

Evaluate the following expressions being sure to follow the order of operations:

5. When $x = 3$,

(a) x^5 (b) 5^x (c) $3x^2 - 4x - 8$

6. When $x = 3$, $y = -2$

$6x^2 + 3xy - 9y^2$

7. When $x = -8$, $y = 3$

$(x + y)^2$

Simplify by combining like terms:

8. $10a + 7 + 5a - 2 + 7a - 4$

9. $5b + 9b + 10(2b + 3b) + 5$

Objective 2: Solve linear equations using a general strategy (IA 2.1.1)**HOW TO**

Solve linear equations using a general strategy

- Step 1. Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
- Step 2. Collect all the variable terms on one side of the equation. Use the Addition or Subtraction Property of Equality.
- Step 3. Collect all the constant terms on the other side of the equation. Use the Addition or Subtraction Property of Equality.
- Step 4. Make the coefficient of the variable term equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
- Step 5. Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2

Solve linear equations using a general strategy.

Solve for w

$$2(w + 5) + 1 = 10 + 4w + 2$$

✔ **Solution**

Use distributive property to remove parentheses:	$2w + 10 + 1 = 10 + 4w + 2$
Combine like terms on each side:	$2w + 11 = 12 + 4w$
Subtract $2w$ from each side to bring variables to one side:	$2w - 2w + 11 = 12 + 4w - 2w$
Combine like terms:	$11 = 12 + 2w$
Subtract 12 from each side to bring constants to one side:	$11 - 12 = 12 - 12 + 2w$
Combine like terms:	$-1 = 2w$
Divide each side by 2 to isolate the variable terms:	$\frac{-1}{2} = \frac{2w}{2}$
Simplify:	$\frac{-1}{2} = w$ or $w = -\frac{1}{2}$
To check your solution, replace w with $-\frac{1}{2}$ in the original equation and simplify:	$2(w + 5) + 1 = 10 + 4w + 2$
	$2(-\frac{1}{2} + 5) + 1 = 10 + 4(-\frac{1}{2}) + 2$
	$2(\frac{9}{2}) + 1 = 10 + (-2) + 2$ $9 + 1 = 8 + 2$
The solution checks, we reached a true statement.	$10 = 10$

Practice Makes Perfect

Solve each linear equation using the general strategy.

10. $15(y - 9) = -60$

11. $-2(11 - 7x) + 54 = 4$

12. $3(4n - 1) - 2 = 8n + 3$

13. $12 + 2(5 - 3y) = -9(y - 1) - 2$

14. $\frac{1}{4}(20x + 12) = x + 7$

15. $22(3m - 4) = 8(2m + 9)$

16. $\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

17. $0.05n + 0.10(n + 8) = 2.15$

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has

taken a part-time job at the local bank that pays \$15.00/hr, and she opened a savings account with an initial deposit of \$400 on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately \$2,500, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in [Figure 1](#).

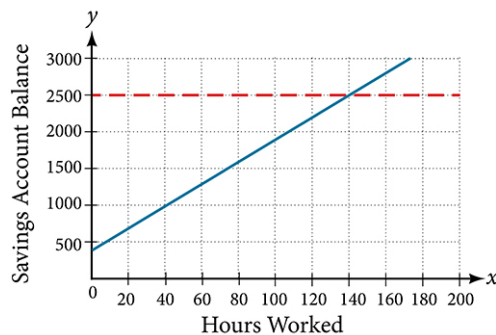


Figure 1

Solving Linear Equations in One Variable

A **linear equation** is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax + b = 0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent. An **identity equation** is true for all values of the variable. Here is an example of an identity equation.

$$3x = 2x + x$$

The **solution set** consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$\begin{aligned} 5x + 2 &= 3x - 6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The solution set consists of one number: $\{-4\}$. It is the only solution and, therefore, we have solved a conditional equation.

An **inconsistent equation** results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$\begin{aligned} 5x - 15 &= 5x - 20 \\ 5x - 15 - 5x &= 5x - 20 - 5x && \text{Subtract } 5x \text{ from both sides.} \\ -15 &\neq -20 && \text{False statement} \end{aligned}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

Linear Equation in One Variable

A linear equation in one variable can be written in the form

$$ax + b = 0$$

where a and b are real numbers, $a \neq 0$.

**HOW TO****Given a linear equation in one variable, use algebra to solve it.**

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x = \underline{\hspace{2cm}}$, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a(b + c) = ab + ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

EXAMPLE 1**Solving an Equation in One Variable**

Solve the following equation: $2x + 7 = 19$.

✓ Solution

This equation can be written in the form $ax + b = 0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$\begin{array}{rcl} 2x + 7 & = & 19 \\ 2x & = & 12 \quad \text{Subtract 7 from both sides.} \\ x & = & 6 \quad \text{Multiply both sides by } \frac{1}{2} \text{ or divide by 2.} \end{array}$$

The solution is 6.



TRY IT #1 Solve the linear equation in one variable: $2x + 1 = -9$.

EXAMPLE 2**Solving an Equation Algebraically When the Variable Appears on Both Sides**

Solve the following equation: $4(x-3) + 12 = 15 - 5(x+6)$.

✓ Solution

Apply standard algebraic properties.

$$\begin{array}{rcl} 4(x-3) + 12 & = & 15 - 5(x+6) \\ 4x - 12 + 12 & = & 15 - 5x - 30 \quad \text{Apply the distributive property.} \\ 4x & = & -15 - 5x \quad \text{Combine like terms.} \\ 9x & = & -15 \quad \text{Place } x\text{-terms on one side and simplify.} \\ x & = & -\frac{15}{9} \quad \text{Multiply both sides by } \frac{1}{9}, \text{ the reciprocal of 9.} \\ x & = & -\frac{5}{3} \end{array}$$

⊙ Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x = -\frac{5}{3}$.



TRY IT #2 Solve the equation in one variable: $-2(3x - 1) + x = 14 - x$.

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is considered a **rational equation**.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}, \quad \frac{1}{x-3}, \quad \text{or} \quad \frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

EXAMPLE 3

Solving a Rational Equation

Solve the rational equation: $\frac{7}{2x} - \frac{5}{3x} = \frac{22}{3}$.

✓ Solution

We have three denominators; $2x$, $3x$, and 3 . The LCD must contain $2x$, $3x$, and 3 . An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$\begin{aligned} (6x) \left(\frac{7}{2x} - \frac{5}{3x} \right) &= \left(\frac{22}{3} \right) (6x) \\ (6x) \left(\frac{7}{2x} \right) - (6x) \left(\frac{5}{3x} \right) &= \left(\frac{22}{3} \right) (6x) && \text{Use the distributive property.} \\ \cancel{(6x)} \left(\frac{7}{\cancel{2x}} \right) - \cancel{(6x)} \left(\frac{5}{\cancel{3x}} \right) &= \left(\frac{22}{\cancel{3}} \right) (\cancel{6}x) && \text{Cancel out the common factors.} \\ 3(7) - 2(5) &= 22(2x) && \text{Multiply remaining factors by each numerator.} \\ 21 - 10 &= 44x \\ 11 &= 44x \\ \frac{11}{44} &= x \\ \frac{1}{4} &= x \end{aligned}$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x , $x-1$, and $3x-3$. First, factor all denominators. We then have x , $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x , one factor of $(x-1)$, and the 3 . Thus, the LCD is the following:

$$x(x-1)3 = 3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x . Once the second denominator is factored as $x^2+2x = x(x+2)$, there is a common factor of x in both denominators and the LCD is $x(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b} = \frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} \times \frac{c}{d}.$$

Multiply $a(d)$ and $b(c)$, which results in $ad = bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

Rational Equations

A **rational equation** contains at least one rational expression where the variable appears in at least one of the denominators.



HOW TO

Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.

EXAMPLE 4

Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

✓ Solution

We have three denominators: x , 2 , and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0 . Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x \left(\frac{2}{x} - \frac{3}{2} \right) &= \left(\frac{7}{2x} \right) 2x \\ 2 \cancel{x} \left(\frac{2}{\cancel{x}} \right) - \cancel{2} x \left(\frac{3}{\cancel{2}} \right) &= \left(\frac{7}{\cancel{2x}} \right) \cancel{2x} && \text{Distribute } 2x. \\ 2(2) - 3x &= 7 && \text{Denominators cancel out.} \\ 4 - 3x &= 7 \\ -3x &= 3 \\ x &= -1 \\ &\text{or } \{-1\} \end{aligned}$$

The proposed solution is -1 , which is not an excluded value, so the solution set contains one number, -1 , or $\{-1\}$ written in set notation.

TRY IT #3 Solve the rational equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$.

EXAMPLE 5**Solving a Rational Equation by Factoring the Denominator**Solve the following rational equation: $\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$.**✓ Solution**

First find the common denominator. The three denominators in factored form are x , $10 = 2 \cdot 5$, and $4x = 2 \cdot 2 \cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x = 0$ is an excluded value. Multiply the whole equation by $20x$.

$$\begin{aligned} 20x \left(\frac{1}{x} \right) &= \left(\frac{1}{10} - \frac{3}{4x} \right) 20x \\ 20 &= 2x - 15 \\ 35 &= 2x \\ \frac{35}{2} &= x \end{aligned}$$

The solution is $\frac{35}{2}$.

> TRY IT #4 Solve the rational equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$.

EXAMPLE 6**Solving Rational Equations with a Binomial in the Denominator**

Solve the following rational equations and state the excluded values:

a $\frac{3}{x-6} = \frac{5}{x}$ **b** $\frac{x}{x-3} = \frac{5}{x-3} - \frac{1}{2}$ **c** $\frac{x}{x-2} = \frac{5}{x-2} - \frac{1}{2}$

✓ Solution**a**

The denominators x and $x - 6$ have nothing in common. Therefore, the LCD is the product $x(x - 6)$. However, for this problem, we can cross-multiply.

$$\begin{aligned} \frac{3}{x-6} &= \frac{5}{x} \\ 3x &= 5(x-6) && \text{Distribute.} \\ 3x &= 5x - 30 \\ -2x &= -30 \\ x &= 15 \end{aligned}$$

The solution is 15. The excluded values are 6 and 0.

bThe LCD is $2(x - 3)$. Multiply both sides of the equation by $2(x - 3)$.

$$\begin{aligned} 2(x-3) \left(\frac{x}{x-3} \right) &= \left(\frac{5}{x-3} - \frac{1}{2} \right) 2(x-3) \\ \frac{\cancel{2(x-3)} x}{\cancel{x-3}} &= \frac{\cancel{2(x-3)} 5}{\cancel{x-3}} - \cancel{2(x-3)} \\ 2x &= 10 - (x-3) \\ 2x &= 10 - x + 3 \\ 2x &= 13 - x \\ 3x &= 13 \\ x &= \frac{13}{3} \end{aligned}$$

The solution is $\frac{13}{3}$. The excluded value is 3.

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The least common denominator is $2(x - 2)$. Multiply both sides of the equation by $x(x - 2)$.

$$\begin{aligned} 2(x - 2) \left(\frac{x}{x - 2} \right) &= \left(\frac{5}{x - 2} - \frac{1}{2} \right) 2(x - 2) \\ 2x &= 10 - (x - 2) \\ 2x &= 12 - x \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

The solution is 4. The excluded value is 2.

> **TRY IT** #5 Solve $\frac{-3}{2x+1} = \frac{4}{3x+1}$. State the excluded values.

EXAMPLE 7

Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

✓ Solution

We must factor the denominator $x^2 - 1$. We recognize this as the difference of squares, and factor it as $(x - 1)(x + 1)$. Thus, the LCD that contains each denominator is $(x - 1)(x + 1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$\begin{aligned} (x - 1)(x + 1) \left(\frac{2}{x + 1} - \frac{1}{x - 1} \right) &= \left(\frac{2x}{(x - 1)(x + 1)} \right) (x - 1)(x + 1) \\ 2(x - 1) - 1(x + 1) &= 2x \\ 2x - 2 - x - 1 &= 2x && \text{Distribute the negative sign.} \\ -3 - x &= 0 \\ -3 &= x \end{aligned}$$

The solution is -3 . The excluded values are 1 and -1 .

> **TRY IT** #6 Solve the rational equation: $\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$.

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as $y = mx + b$, where $m =$ slope and $b =$ y -intercept. Let us begin with the slope.

The Slope of a Line

The **slope** of a line refers to the ratio of the vertical change in y over the horizontal change in x between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in [Figure 2](#). The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

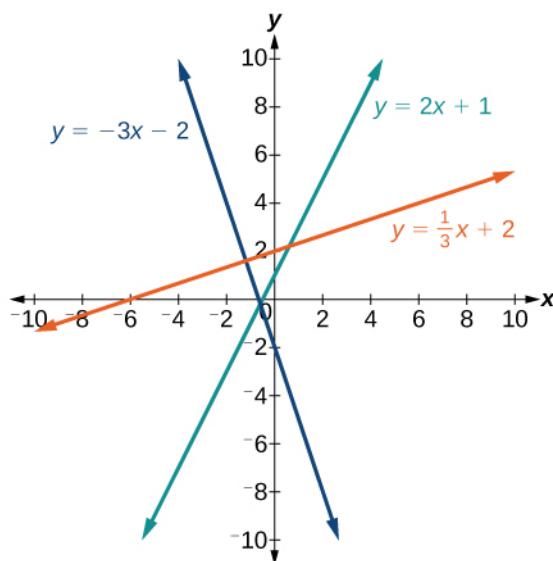


Figure 2

The Slope of a Line

The slope of a line, m , represents the change in y over the change in x . Given two points, (x_1, y_1) and (x_2, y_2) , the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 8

Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

✓ Solution

We substitute the y -values and the x -values into the formula.

$$\begin{aligned} m &= \frac{3 - (-1)}{-5 - 2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

🔍 Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2) . As long as we are consistent with the order of the y terms and the order of the x terms in the numerator and denominator, the calculation will yield the same result.

> **TRY IT #7** Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

EXAMPLE 9

Identifying the Slope and y -intercept of a Line Given an Equation

Identify the slope and y -intercept, given the equation $y = -\frac{3}{4}x - 4$.

✓ Solution

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The y -intercept is $b = -4$.

⊙ Analysis

The y -intercept is the point at which the line crosses the y -axis. On the y -axis, $x = 0$. We can always identify the y -intercept when the line is in slope-intercept form, as it will always equal b . Or, just substitute $x = 0$ and solve for y .

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

$$y - y_1 = m(x - x_1)$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

The Point-Slope Formula

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1)$$

EXAMPLE 10**Finding the Equation of a Line Given the Slope and One Point**

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

✓ Solution

Using the point-slope formula, substitute -3 for m and the point $(4, 8)$ for (x_1, y_1) .

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

⊙ Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

> TRY IT #8 Given $m = 4$, find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

EXAMPLE 11**Finding the Equation of a Line Passing Through Two Given Points**

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

✓ Solution

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned} m &= \frac{-3-4}{0-3} \\ &= \frac{-7}{-3} \\ &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point (3, 4) for (x_1, y_1) .

$$\begin{aligned}y - 4 &= \frac{7}{3}(x - 3) \\y - 4 &= \frac{7}{3}x - 7 \quad \text{Distribute the } \frac{7}{3}. \\y &= \frac{7}{3}x - 3\end{aligned}$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$.

Analysis

To prove that either point can be used, let us use the second point (0, -3) and see if we get the same equation.

$$\begin{aligned}y - (-3) &= \frac{7}{3}(x - 0) \\y + 3 &= \frac{7}{3}x \\y &= \frac{7}{3}x - 3\end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in standard form. Standard form is given as

$$Ax + By = C$$

where A , B , and C are integers. The x - and y -terms are on one side of the equal sign and the constant term is on the other side.

EXAMPLE 12

Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $(\frac{1}{4}, -2)$. Write the equation in standard form.

Solution


We begin using the point-slope formula.

$$\begin{aligned}y - (-2) &= -6\left(x - \frac{1}{4}\right) \\y + 2 &= -6x + \frac{3}{2}\end{aligned}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left side of the equal sign and move the constants to the right.

$$\begin{aligned}2(y + 2) &= \left(-6x + \frac{3}{2}\right)2 \\2y + 4 &= -12x + 3 \\12x + 2y &= -1\end{aligned}$$

This equation is now written in standard form.

 **TRY IT** #9 Find the equation of the line in standard form with slope $m = -\frac{1}{3}$ and passing through the point $(1, \frac{1}{3})$.

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as

$$x = c$$

where c is a constant. The slope of a vertical line is undefined, and regardless of the y -value of any point on the line, the x -coordinate of the point will be c .

Suppose that we want to find the equation of a line containing the following points: $(-3, -5)$, $(-3, 1)$, $(-3, 3)$, and $(-3, 5)$. First, we will find the slope.

$$m = \frac{5 - 3}{-3 - (-3)} = \frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the x -coordinates are the same and we find a vertical line through $x = -3$. See [Figure 3](#).

The equation of a horizontal line is given as

$$y = c$$

where c is a constant. The slope of a horizontal line is zero, and for any x -value of a point on the line, the y -coordinate will be c .

Suppose we want to find the equation of a line that contains the following set of points: $(-2, -2)$, $(0, -2)$, $(3, -2)$, and $(5, -2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2 - (-2)}{0 - (-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for (x_1, y_1) in the formula, or use the y -intercept.

$$\begin{aligned} y - (-2) &= 0(x - 3) \\ y + 2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y = -2$. Notice that all of the y -coordinates are the same. See [Figure 3](#).

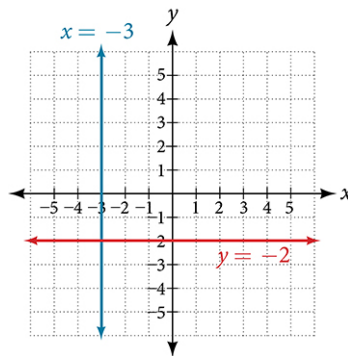


Figure 3 The line $x = -3$ is a vertical line. The line $y = -2$ is a horizontal line.

EXAMPLE 13

Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1, -3)$ and $(1, 4)$.

✓ Solution

The x -coordinate of both points is 1. Therefore, we have a vertical line, $x = 1$.

> **TRY IT** #10 Find the equation of the line passing through $(-5, 2)$ and $(2, 2)$.

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different y -intercepts. Lines that are parallel to each other will never intersect. For example, [Figure 4](#) shows the graphs of various lines with the same slope, $m = 2$.

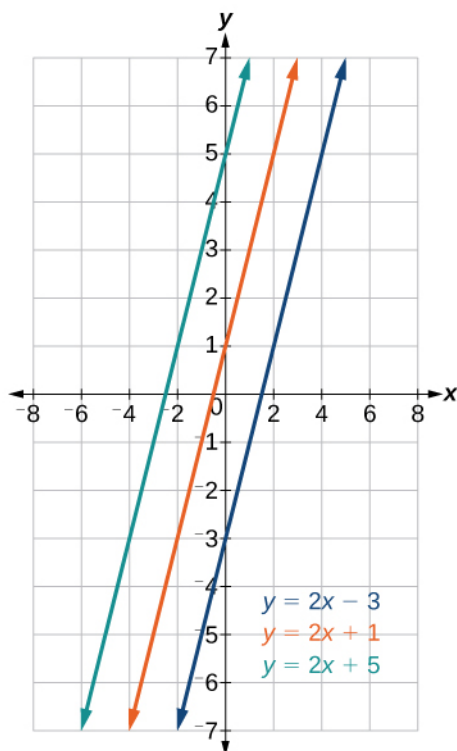


Figure 4 Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different y -intercepts.

Lines that are perpendicular intersect to form a 90° -angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are perpendicular if the product of the two slopes is -1 : $m_1 \cdot m_2 = -1$. For example, [Figure 5](#) shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

$$m_1 \cdot m_2 = -1$$

$$3 \cdot \left(-\frac{1}{3}\right) = -1$$

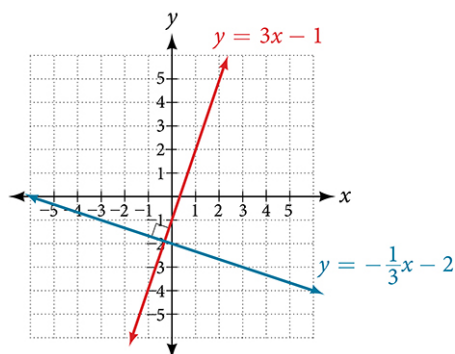


Figure 5 Perpendicular lines

EXAMPLE 14**Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither**

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y = -4x + 3$ and $3x - 4y = 8$.

✓ Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$3y = -4x + 3$$

$$y = -\frac{4}{3}x + 1$$

Second equation:

$$3x - 4y = 8$$

$$-4y = -3x + 8$$

$$y = \frac{3}{4}x - 2$$

See the graph of both lines in [Figure 6](#)

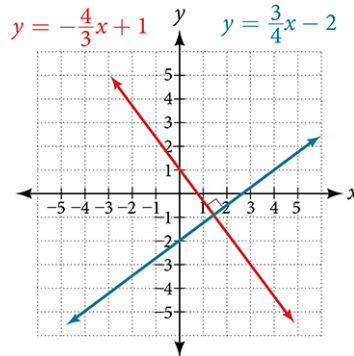


Figure 6

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$m_1 = -\frac{4}{3}$$

$$m_2 = \frac{3}{4}$$

$$m_1 \cdot m_2 = \left(-\frac{4}{3}\right)\left(\frac{3}{4}\right) = -1$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

> **TRY IT** #11 Graph the two lines and determine whether they are parallel, perpendicular, or neither:
 $2y - x = 10$ and $2y = x + 4$.

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the point-slope formula to write the equation of the new line.



HOW TO

Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

EXAMPLE 15

Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

✔ **Solution**

First, we will write the equation in slope-intercept form to find the slope.

$$\begin{aligned} 5x + 3y &= 1 \\ 3y &= -5x + 1 \\ y &= -\frac{5}{3}x + \frac{1}{3} \end{aligned}$$

The slope is $m = -\frac{5}{3}$. The y -intercept is $\frac{1}{3}$, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y -intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$\begin{aligned} y - 5 &= -\frac{5}{3}(x - 3) \\ y - 5 &= -\frac{5}{3}x + 5 \\ y &= -\frac{5}{3}x + 10 \end{aligned}$$

The equation of the line is $y = -\frac{5}{3}x + 10$. See [Figure 7](#).

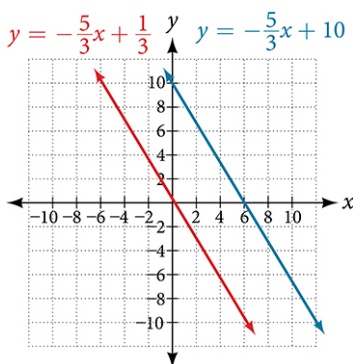


Figure 7

> **TRY IT** #12 Find the equation of the line parallel to $5x = 7 + y$ and passing through the point $(-1, -2)$.

EXAMPLE 16

Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x - 3y + 4 = 0$ and passing through the point $(-4, 1)$.

✔ **Solution**

The first step is to write the equation in slope-intercept form.

$$\begin{aligned} 5x - 3y + 4 &= 0 \\ -3y &= -5x - 4 \\ y &= \frac{5}{3}x + \frac{4}{3} \end{aligned}$$

We see that the slope is $m = \frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$\begin{aligned} y - 1 &= -\frac{3}{5}(x - (-4)) \\ y - 1 &= -\frac{3}{5}x - \frac{12}{5} \\ y &= -\frac{3}{5}x - \frac{12}{5} + \frac{5}{5} \\ y &= -\frac{3}{5}x - \frac{7}{5} \end{aligned}$$

 **MEDIA**

Access these online resources for additional instruction and practice with linear equations.

[Solving rational equations \(http://openstax.org//rationaleqs\)](http://openstax.org//rationaleqs)

[Equation of a line given two points \(http://openstax.org//twopointsline\)](http://openstax.org//twopointsline)

[Finding the equation of a line perpendicular to another line through a given point \(http://openstax.org//findperpline\)](http://openstax.org//findperpline)

[Finding the equation of a line parallel to another line through a given point \(http://openstax.org//findparaline\)](http://openstax.org//findparaline)



2.2 SECTION EXERCISES

Verbal

- What does it mean when we say that two lines are parallel?
- What is the relationship between the slopes of perpendicular lines (assuming neither is horizontal nor vertical)?
- How do we recognize when an equation, for example $y = 4x + 3$, will be a straight line (linear) when graphed?
- What does it mean when we say that a linear equation is inconsistent?
- When solving the following equation:

$$\frac{2}{x-5} = \frac{4}{x+1}$$
 explain why we must exclude $x = 5$ and $x = -1$ as possible solutions from the solution set.

Algebraic

For the following exercises, solve the equation for x .

- $7x + 2 = 3x - 9$
- $4x - 3 = 5$
- $3(x + 2) - 12 = 5(x + 1)$
- $12 - 5(x + 3) = 2x - 5$
- $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$
- $\frac{x}{3} - \frac{3}{4} = \frac{2x+3}{12}$
- $\frac{2}{3}x + \frac{1}{2} = \frac{31}{6}$
- $3(2x - 1) + x = 5x + 3$
- $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$
- $\frac{x+2}{4} - \frac{x-1}{3} = 2$

For the following exercises, solve each rational equation for x . State all x -values that are excluded from the solution set.

- $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$
- $2 - \frac{3}{x+4} = \frac{x+2}{x+4}$
- $\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$
- $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$
- $\frac{5}{x+1} + \frac{1}{x-3} = \frac{-6}{x^2-2x-3}$
- $\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$

For the following exercises, find the equation of the line using the point-slope formula. Write all the final equations using the slope-intercept form.

22. $(0, 3)$ with a slope of $\frac{2}{3}$ 23. $(1, 2)$ with a slope of $-\frac{4}{5}$ 24. x-intercept is 1, and $(-2, 6)$
25. y-intercept is 2, and $(4, -1)$ 26. $(-3, 10)$ and $(5, -6)$ 27. $(1, 3)$ and $(5, 5)$
28. parallel to $y = 2x + 5$ and passes through the point $(4, 3)$ 29. perpendicular to $3y = x - 4$ and passes through the point $(-2, 1)$.

For the following exercises, find the equation of the line using the given information.

30. $(-2, 0)$ and $(-2, 5)$ 31. $(1, 7)$ and $(3, 7)$ 32. The slope is undefined and it passes through the point $(2, 3)$.
33. The slope equals zero and it passes through the point $(1, -4)$. 34. The slope is $\frac{3}{4}$ and it passes through the point $(1, 4)$. 35. $(-1, 3)$ and $(4, -5)$

Graphical

For the following exercises, graph the pair of equations on the same axes, and state whether they are parallel, perpendicular, or neither.

36. $y = 2x + 7$
 $y = -\frac{1}{2}x - 4$ 37. $3x - 2y = 5$
 $6y - 9x = 6$ 38. $y = \frac{3x+1}{4}$
 $y = 3x + 2$
39. $x = 4$
 $y = -3$

Numeric

For the following exercises, find the slope of the line that passes through the given points.

40. $(5, 4)$ and $(7, 9)$ 41. $(-3, 2)$ and $(4, -7)$ 42. $(-5, 4)$ and $(2, 4)$
43. $(-1, -2)$ and $(3, 4)$ 44. $(3, -2)$ and $(3, -2)$

For the following exercises, find the slope of the lines that pass through each pair of points and determine whether the lines are parallel or perpendicular.

45. $(-1, 3)$ and $(5, 1)$
 $(-2, 3)$ and $(0, 9)$ 46. $(2, 5)$ and $(5, 9)$
 $(-1, -1)$ and $(2, 3)$

Technology

For the following exercises, express the equations in slope intercept form (rounding each number to the thousandths place). Enter this into a graphing calculator as Y1, then adjust the ymin and ymax values for your window to include where the y-intercept occurs. State your ymin and ymax values.

47. $0.537x - 2.19y = 100$

48. $4,500x - 200y = 9,528$

49. $\frac{200-30y}{x} = 70$

Extensions

50. Starting with the point-slope formula $y - y_1 = m(x - x_1)$, solve this expression for x in terms of x_1 , y , y_1 , and m .

51. Starting with the standard form of an equation $Ax + By = C$ solve this expression for y in terms of A , B , C and x . Then put the expression in slope-intercept form.

52. Use the above derived formula to put the following standard equation in slope intercept form: $7x - 5y = 25$.

53. Given that the following coordinates are the vertices of a rectangle, prove that this truly is a rectangle by showing the slopes of the sides that meet are perpendicular.

$(-1, 1)$, $(2, 0)$, $(3, 3)$ and $(0, 4)$

54. Find the slopes of the diagonals in the previous exercise. Are they perpendicular?

Real-World Applications

55. The slope for a wheelchair ramp for a home has to be $\frac{1}{12}$. If the vertical distance from the ground to the door bottom is 2.5 ft, find the distance the ramp has to extend from the home in order to comply with the needed slope.



56. If the profit equation for a small business selling x number of item one and y number of item two is $p = 3x + 4y$, find the y value when $p = \$453$ and $x = 75$.

For the following exercises, use this scenario: The cost of renting a car is \$45/wk plus \$0.25/mi traveled during that week. An equation to represent the cost would be $y = 45 + .25x$, where x is the number of miles traveled.

57. What is your cost if you travel 50 mi?
58. If your cost were \$63.75, how many miles were you charged for traveling?
59. Suppose you have a maximum of \$100 to spend for the car rental. What would be the maximum number of miles you could travel?

4.4 composition of functions

4.4.1 composite functions (Active Prelude to Calculus)

4.4.2 composition of functions and Chain Rule (TBIL)

4.4.3 composite functions and differentiation strategies (TBIL)

1.6 Composite Functions

Motivating Questions

- How does the process of function composition produce a new function from two other functions?
 - In the composite function $h(x) = f(g(x))$, what do we mean by the “inner” and “outer” function? What role do the domain and codomain of f and g play in determining the domain and codomain of h ?
 - How does the expression for $AV_{[a,a+h]}$ involve a composite function?
-

Recall that a function, by definition, is a process that takes a collection of inputs and produces a corresponding collection of outputs in such a way that the process produces one and only one output value for any single input value. Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 1.6.1 Suppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t ?

Solution. Since y depends on x and x depends on t , it follows that we can also think of y depending directly on t . We can use substitution and the notation of functions to determine this relationship.

First, it's important to realize what the rule for f tells us. In words, f says “to generate the output that corresponds to an input, take the input and square it, and then subtract 1.” In symbols, we might express f more generally by writing “ $f(\square) = \square^2 - 1$.”

Now, observing that $y = f(x) = x^2 - 1$ and that $x = g(t) = 3t - 4$, we can substitute the expression $g(t)$ for x in f . Doing so,

$$\begin{aligned} y &= f(x) \\ &= f(g(t)) \\ &= f(3t - 4). \end{aligned}$$

Applying the process defined by the function f to the input $3t - 4$, we see that

$$y = (3t - 4)^2 - 1,$$

which defines y as a function of t . □

When we have a situation such as in Example 1.6.1 where we use the output of one function as the input of another, we often say that we have “composed two functions”. In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h , results from composing the two functions f and g .

Preview Activity 1.6.1. Let $y = p(x) = 3x - 4$ and $x = q(t) = t^2 - 1$.

- Let $r(t) = p(q(t))$. Determine a formula for r that depends only on t and not on p or q .
- Recall Example 1.6.1, which involved functions similar to p and q . What is the biggest difference between your work in (a) above and in Example 1.6.1?
- Let $t = s(z) = \frac{1}{z+4}$ and recall that $x = q(t) = t^2 - 1$. Determine a formula for $x = q(s(z))$ that depends only on z .
- Suppose that $h(t) = \sqrt{2t^2 + 5}$. Determine formulas for two related functions, $y = f(x)$ and $x = g(t)$, so that $h(t) = f(g(t))$.

1.6.1 Composing two functions

Whenever we have two functions, say $g : A \rightarrow B$ and $f : B \rightarrow C$, where the codomain of g matches the domain of f , it is possible to link the two processes together to create a new process that we call the *composition* of f and g .

Definition 1.6.2 If f and g are functions such that $g : A \rightarrow B$ and $f : B \rightarrow C$, we define the **composition of f and g** to be the new function $h : A \rightarrow C$ given by

$$h(t) = f(g(t)).$$

We also sometimes use the notation $h = f \circ g$, where $f \circ g$ is the single function defined by $(f \circ g)(t) = f(g(t))$. \diamond

We sometimes call g the “inner function” and f the “outer function”. It is important to note that the inner function is actually the first function that gets applied to a given input, and then outer function is applied to the output of the inner function. In addition, in order for a composite function to make sense, we need to ensure that the range of the inner function lies within the domain of the outer function so that the resulting composite function is defined at every possible input.

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Activity 1.6.2. Let functions p and q be given by the graphs in Figure 1.6.4 (which are each piecewise linear - that is, parts that look like straight lines are straight lines) and let f and g be given by Table 1.6.3.

x	0	1	2	3	4
$f(x)$	6	4	3	4	6
$g(x)$	1	3	0	4	2

Table 1.6.3: Table that defines f and g .

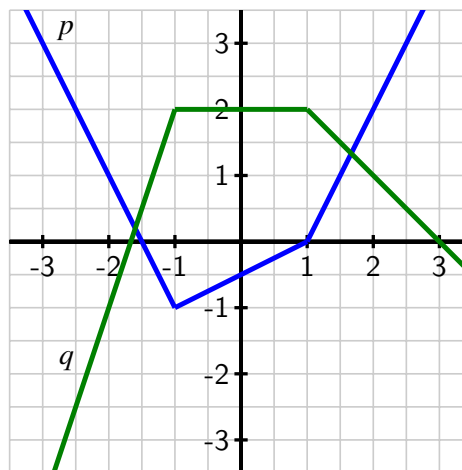


Figure 1.6.4: The graphs of p and q .

Compute each of the following quantities or explain why they are not defined.

- $p(q(0))$
- $q(p(0))$
- $(p \circ p)(-1)$
- $(f \circ g)(2)$
- $(g \circ f)(3)$
- $q(f(0))$
- For what value(s) of x is $f(g(x)) = 4$?
- For what value(s) of x is $q(p(x)) = 1$?

1.6.2 Composing functions in context

Recall Dolbear's function, $T = D(N) = 40 + 0.25N$, that relates the number of chirps per minute from a snowy cricket to the Fahrenheit temperature, T . We earlier established that D has a domain of $[40, 160]$ and a corresponding range of $[50, 85]$. In what follows, we replace T with F to emphasize that temperature is measured in Fahrenheit degrees.

The Celcius and Fahrenheit temperature scales are connected by a linear function. Indeed, the function that converts Fahrenheit to Celcius is

$$C = G(F) = \frac{5}{9}(F - 32).$$

For instance, a Fahrenheit temperature of 32 degrees corresponds to $C = G(32) = 0$ degrees Celcius.

Activity 1.6.3. Let $F = D(N) = 40 + 0.25N$ be Dolbear's function that converts an input of number of chirps per minute to degrees Fahrenheit, and let $C = G(F) = \frac{5}{9}(F - 32)$ be the function that converts an input of degrees Fahrenheit to an output of degrees Celsius.

- Determine a formula for the new function $H = (G \circ D)$ that depends only on the variable N .
- What is the meaning of the function you found in (a)?
- How does a plot of the function $H = (G \circ D)$ compare to that of Dolbear's function? Sketch a plot of $y = H(N) = (G \circ D)(N)$ on the blank axes to the right of the plot of Dolbear's function, and discuss the similarities and differences between them. Be sure to label the vertical scale on your axes.

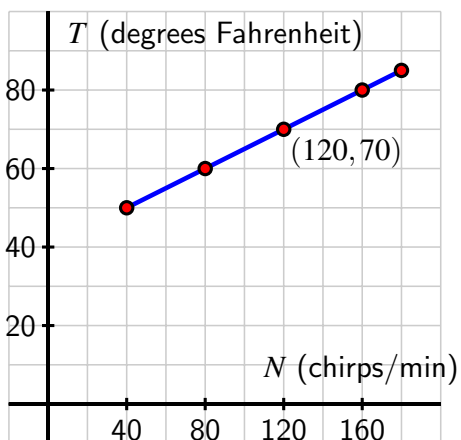


Figure 1.6.5: Dolbear's function.

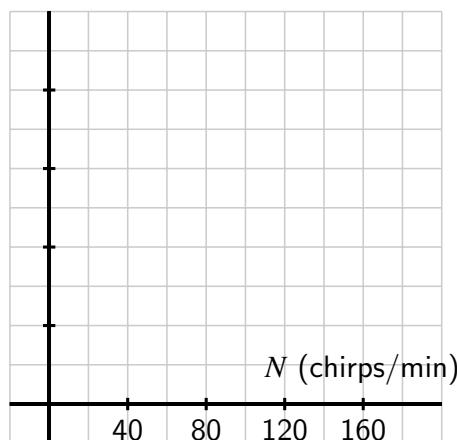


Figure 1.6.6: Blank axes to plot $H = (G \circ D)(N)$.

- What is the domain of the function $H = G \circ D$? What is its range?

1.6.3 Function composition and average rate of change

Recall that the average rate of change of a function f on the interval $[a, b]$ is given by

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

In Figure 1.6.7, we see the familiar representation of $AV_{[a,b]}$ as the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f . In the study of calculus, we progress from the *average rate of change on an interval* to the *instantaneous rate of change of a function at a single value*; the core idea that allows us to move from an *average* rate to an *instantaneous* one is letting the interval $[a, b]$ shrink in size.

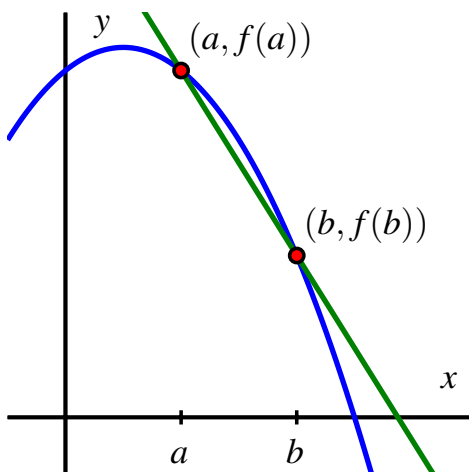


Figure 1.6.7: $AV_{[a,b]}$ is the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f .

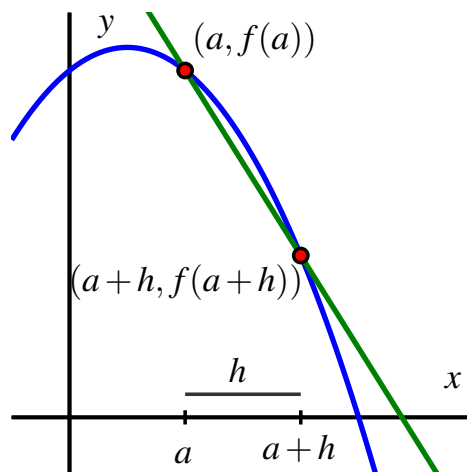


Figure 1.6.8: $AV_{[a,a+h]}$ is the slope of the line joining the points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph of f .

To think about the interval $[a, b]$ shrinking while a stays fixed, we often change our perspective and think of b as $b = a + h$, where h measures the horizontal difference from b to a . This allows us to eventually think about h getting closer and closer to 0, and in that context we consider the equivalent expression

$$AV_{[a,a+h]} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

for the average rate of change of f on $[a, a+h]$.

In this most recent expression for $AV_{[a,a+h]}$, we see the important role that the composite function " $f(a+h)$ " plays. In particular, to understand the expression for $AV_{[a,a+h]}$ we need to evaluate f at the quantity $(a+h)$.

Example 1.6.9 Suppose that $f(x) = x^2$. Determine the simplest possible expression you can find for $AV_{[3,3+h]}$, the average rate of change of f on the interval $[3, 3+h]$.

Solution. By definition, we know that

$$AV_{[3,3+h]} = \frac{f(3+h) - f(3)}{h}.$$

Using the formula for f , we see that

$$AV_{[3,3+h]} = \frac{(3+h)^2 - (3)^2}{h}.$$

Expanding the numerator and combining like terms, it follows that

$$AV_{[3,3+h]} = \frac{(9 + 6h + h^2) - 9}{h}$$

$$= \frac{6h + h^2}{h}.$$

Removing a factor of h in the numerator and observing that $h \neq 0$, we can simplify and find that

$$\begin{aligned} AV_{[3,3+h]} &= \frac{h(6+h)}{h} \\ &= 6+h. \end{aligned}$$

Hence, $AV_{[3,3+h]} = 6+h$, which is the average rate of change of $f(x) = x^2$ on the interval $[3, 3+h]$.¹ □

Activity 1.6.4. Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = \frac{5}{x}$.

- Compute $f(1+h)$ and expand and simplify the result as much as possible by combining like terms.
- Determine the most simplified expression you can for the average rate of change of f on the interval $[1, 1+h]$. That is, determine $AV_{[1,1+h]}$ for f and simplify the result as much as possible.
- Compute $g(1+h)$. Is there any valid algebra you can do to write $g(1+h)$ more simply?
- Determine the most simplified expression you can for the average rate of change of g on the interval $[1, 1+h]$. That is, determine $AV_{[1,1+h]}$ for g and simplify the result.

In Activity 1.6.4, we see an important setting where algebraic simplification plays a crucial role in calculus. Because the expression

$$AV_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$

always begins with an h in the denominator, in order to precisely understand how this quantity behaves when h gets close to 0, a simplified version of this expression is needed. For instance, as we found in part (b) of Activity 1.6.4, it's possible to show that for $f(x) = 2x^2 - 3x + 1$,

$$AV_{[1,1+h]} = 2h + 1,$$

which is a much simpler expression to investigate.

1.6.4 Summary

- When defined, the composition of two functions f and g produces a single new function $f \circ g$ according to the rule $(f \circ g)(x) = f(g(x))$. We note that g is applied first to the input x , and then f is applied to the output $g(x)$ that results from g .


¹Note that $6+h$ is a *linear* function of h . This computation is connected to the observation we made in Table 1.5.9 regarding how there's a linear aspect to how the average rate of change of a quadratic function changes as we modify the interval.

- In the composite function $h(x) = f(g(x))$, the “inner” function is g and the “outer” function is f . Note that the inner function gets applied to x first, even though the outer function appears first when we read from left to right. The composite function is only defined provided that the codomain of g matches the domain of f : that is, we need any possible outputs of g to be among the allowed inputs for f . In particular, we can say that if $g : A \rightarrow B$ and $f : B \rightarrow C$, then $f \circ g : A \rightarrow C$. Thus, the domain of the composite function is the domain of the inner function, and the codomain of the composite function is the codomain of the outer function.
- Because the expression $AV_{[a,a+h]}$ is defined by

$$AV_{[a,a+h]} = \frac{f(a+h) - f(a)}{h}$$


and this includes the quantity $f(a+h)$, the average rate of change of a function on the interval $[a, a+h]$ always involves the evaluation of a composite function expression. This idea plays a crucial role in the study of calculus.

1.6.5 Exercises

1. Suppose $r = f(t)$ is the radius, in centimeters, of a circle at time t minutes, and $A(r)$ is the area, in square centimeters, of a circle of radius r centimeters. 

Which of the following statements best explains the meaning of the composite function $A(f(t))$?

- The area of a circle, in square centimeters, of radius r centimeters.
- The area of a circle, in square centimeters, at time t minutes.
- The radius of a circle, in centimeters, at time t minutes.
- The function f of the minutes and the area.
- None of the above

2. A swinging pendulum is constructed from a piece of string with a weight attached to the bottom. The length of the pendulum depends on how much string is let out. Suppose $L = f(t)$ is the length, in centimeters, of the pendulum at time t minutes, and $P(L)$ is the period, in seconds, of a pendulum of length L . 

Which of the following statements best explains the meaning of the composite function $P(f(t))$?

- The period P of the pendulum, in minutes, after t minutes have elapsed.
- The period P of the pendulum, in seconds, when the pendulum has length L meters.
- The period P of the pendulum, in minutes, when the pendulum has length L meters.
- The period P of the pendulum, in seconds, after t minutes have elapsed.

⊙ None of the above



3. The formula for the volume of a cube with side length s is $V = s^3$. The formula for the surface area of a cube is $A = 6s^2$.

(a) Find the formula for the function $s = f(A)$.

Which of the statements best explains the meaning of $s = f(A)$?

- ⊙ The side length for a cube of surface area A
- ⊙ The side length for a cube of volume V
- ⊙ The volume of a cube of side length s
- ⊙ The surface area of a cube of side length s

(b) If $V = g(s)$, find a formula for $g(f(A))$.

Which of the statements best explains the meaning of $g(f(A))$?

- ⊙ The volume for a cube of side length s
- ⊙ The surface area for a cube of side length s
- ⊙ The volume for a cube with surface area A
- ⊙ The surface area for a cube of volume V



4. Given that $f(x) = 5x - 6$ and $g(x) = 2x - 2$, calculate

(a) $f \circ g(x) =$ _____

(b) $g \circ f(x) =$ _____

(c) $f \circ f(x) =$ _____

(d) $g \circ g(x) =$ _____



5. This problem gives you some practice identifying how more complicated functions can be built from simpler functions.

Let $f(x) = x^3 - 27$ and let $g(x) = x - 3$. Match the functions defined below with the letters labeling their equivalent expressions.

- | | |
|----------------|------------------------|
| 1. $f(x)/g(x)$ | A. $-27 + x^6$ |
| 2. $f(x^2)$ | B. $9 + 3x + x^2$ |
| 3. $(f(x))^2$ | C. $729 - 54x^3 + x^6$ |
| 4. $g(f(x))$ | D. $-30 + x^3$ |

6. The number of bacteria in a refrigerated food product is given by $N(T) = 27T^2 - 97T + 51$, $3 < T < 33$ where T is the temperature of the food.



When the food is removed from the refrigerator, the temperature is given by $T(t) = 4t + 1.7$, where t is the time in hours. Find the composite function $N(T(t))$.



Find the time when the bacteria count reaches 14225.

7. Let $f(x) = 5x + 2$ and $g(x) = 4x^2 + 3x$. Find $(f \circ g)(-2)$ and $(f \circ g)(x)$.
8. Use the given information about various functions to answer the following questions involving composition.
- a. Let functions f and g be given by the graphs in Figure 1.6.10 and 1.6.11. An open circle means there is not a point at that location on the graph. For instance, $f(-1) = 1$, but $f(3)$ is not defined.

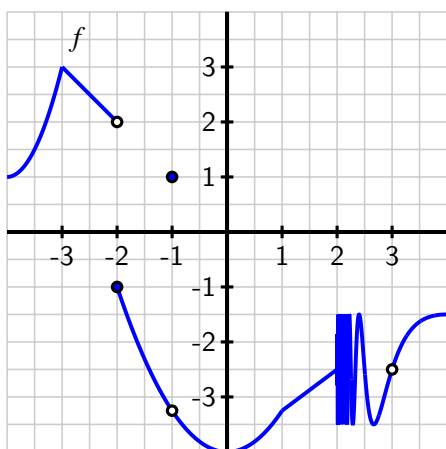


Figure 1.6.10: Plot of $y = f(x)$.

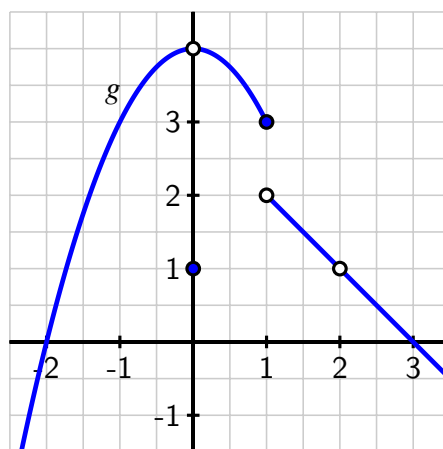


Figure 1.6.11: Plot of $y = g(x)$.

Determine $f(g(1))$ and $g(f(-2))$.

- b. Again using the functions given in (a), can you determine a value of x for which $g(f(x))$ is not defined? Why or why not?
- c. Let functions r and s be defined by Table 1.6.12.

t	-4	-3	-2	-1	0	1	2	3	4
$r(t)$	4	1	2	3	0	-3	2	-1	-4
$s(t)$	-5	-6	-7	-8	0	8	7	6	5

Table 1.6.12: Table that defines r and s .

Determine $(s \circ r)(3)$, $(s \circ r)(-4)$, and $(s \circ r)(a)$ for one additional value of a of your choice.

- d. For the functions r and s defined in (c), state the domain and range of each function. For how many different values of b is it possible to determine $(r \circ s)(b)$? Explain.

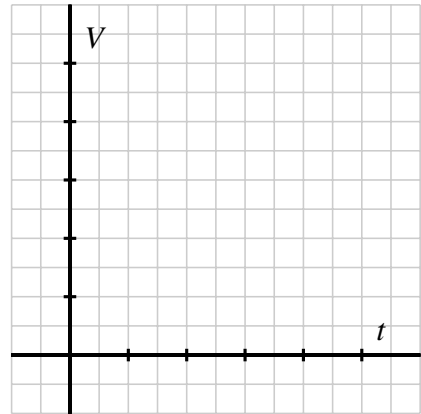
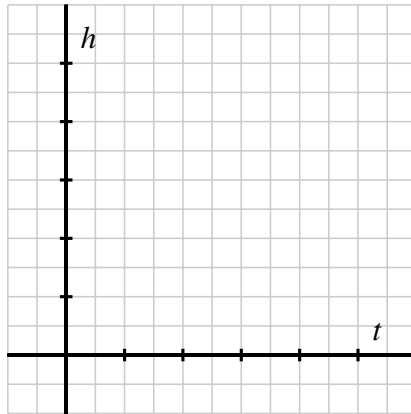
- e. Let $m(u) = u^3 + 4u^2 - 5u + 1$. Determine expressions for $m(x^2)$, $m(2 + h)$, and $m(a + h)$.
- f. For the function $F(x) = 4 - 3x - x^2$, determine the most simplified expression you can find for $AV_{[2,2+h]}$. Show your algebraic work and thinking fully.
9. Recall Dolbear's function that defines temperature, F , in Fahrenheit degrees, as a function of the number of chirps per minute, N , is $F = D(N) = 40 + \frac{1}{4}N$.
- Solve the equation $F = 40 + \frac{1}{4}N$ for N in terms of F .
 - Say that $N = g(F)$ is the function you just found in (a). What is the meaning of this function? What does it take as inputs and what does it produce as outputs?
 - How many chirps per minute do we expect when the outside temperature is 82 degrees F? How can we express this in the notation of the function g ?
 - Recall that the function that converts Fahrenheit to Celsius is $C = G(F) = \frac{5}{9}(F - 32)$. Solve the equation $C = \frac{5}{9}(F - 32)$ for F in terms of C . Call the resulting function $F = p(C)$. What is the meaning of this function?
 - Is it possible to write the chirp-rate N as a function of temperature C in Celsius? That is, can we produce a function whose input is in degrees Celsius and whose output is the number of chirps per minute? If yes, do so and explain your thinking. If not, explain why it's not possible.
10. For each of the following functions, find two simpler functions f and g such that the given function can be written as the composite function $g \circ f$.
- $h(x) = (x^2 + 7)^3$
 - $r(x) = \sqrt{5 - x^3}$
 - $m(x) = \frac{1}{x^4 + 2x^2 + 1}$
 - $w(x) = 2^{3-x^2}$
11. A spherical tank has radius 4 feet. The tank is initially empty and then begins to be filled in such a way that the height of the water rises at a constant rate of 0.4 feet per minute. Let V be the volume of water in the tank at a given instant, and h the depth of the water at the same instant; let t denote the time elapsed in minutes since the tank started being filled.
- Calculus can be used to show that the volume, V , is a function of the depth, h , of the water in the tank according to the function

$$V = f(h) = \frac{\pi}{3}h^2(12 - h). \quad (1.6.1)$$

What is the domain of this model? Why? What is the corresponding range?
 - We are given the fact that the tank is being filled in such a way that the height of the water rises at a constant rate of 0.4 feet per minute. Said differently, h is a function of t whose average rate of change is constant. What kind of function does this make $h = p(t)$? Determine a formula for $p(t)$.
 - What are the domain and range of the function $h = p(t)$? How is this tied to the dimensions of the tank?
 - In (a) we observed that V is a function of h , and in (b) we found that h is a function

of t . Use these two facts and function composition appropriately to write V as a function of t . Call the resulting function $V = q(t)$.

- e. What are the domain and range of the function q ? Why?
- f. On the provided axes, sketch accurate graphs of $h = p(t)$ and $V = q(t)$, labeling the vertical and horizontal scale on each graph appropriately. Make your graphs as precise as you can; use a computing device to assist as needed.



Why do each of the two graphs have their respective shapes? Write at least one sentence to explain each graph; refer explicitly to the shape of the tank and other information given in the problem.

2.4.2 Videos



YouTube: <https://www.youtube.com/watch?v=wqdjFSZe6Dk>

Figure 46 Video for DF4

2.5 The chain rule (DF5)

Learning Outcomes

- Compute derivatives using the Chain Rule.

2.5.1 Activities

Note 2.5.1 When we consider the composition $f \circ g$ of the function f with the function g , we mean the composite function $f(g(x))$, where the function g is applied first and then f is applied to the output of g . We also call f the outside function whilst g is the inside function.

Activity 2.5.2

- (a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Which of the following is a formula for $f(g(x))$?

- | | |
|---------------------|----------------|
| A. $-4x^2 + 4x + 4$ | C. $-2x^2 + 9$ |
| B. $4x^2 - 4x + 5$ | D. $-2x^2 + 4$ |

- (b) One of the options above is a formula for $g(f(x))$. Which one?

Activity 2.5.3

- (a) Consider the composite function $f(g(x)) = \sqrt{e^x}$. Which function is the outside function $f(x)$ and which one is the inside function $g(x)$?

- | | |
|-------------------------------------|-------------------------------------|
| A. $f(x) = x^2$, $g(x) = e^x$ | C. $f(x) = e^x$, $g(x) = \sqrt{x}$ |
| B. $f(x) = \sqrt{x}$, $g(x) = e^x$ | D. $f(x) = e^x$, $g(x) = x^2$ |

- (b) Using properties of exponents, we can rewrite the original function as $e^{\frac{x}{2}}$. Using this new expression, what is your new inside function and your new outside function?

- (c) Consider the function $e^{\sqrt{x}}$. In this case, what are the inside and outside functions?

Activity 2.5.4 In this activity we will build the intuition for the chain rule using a real-world scenario and differential notation for derivatives. Consider the following scenario.

My neighborhood is being invaded! The squirrel population grows based

on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. How fast is the squirrel population growing per week?

- (a) The scenario gives you information regarding the rate of growth of $s(a)$, the squirrel population as a function of acorn availability (measured in bushels). What is the current value of $\frac{ds}{da}$?

A. 2
B. 100
C. 200
D. 50

- (b) The scenario gives you information regarding the rate of growth of $a(t)$, the acorn availability as a function of time (measured in weeks). What is the current value of $\frac{da}{dt}$?

A. 2
B. 100
C. 200
D. 50

- (c) Given all the information provided, what is your best guess for the value of $\frac{ds}{dt}$, the rate at which the squirrel population is growing per week?

A. 2
B. 100
C. 200
D. 50

- (d) Given your answers above, what is the relationship between $\frac{ds}{da}$, $\frac{da}{dt}$, $\frac{ds}{dt}$?

Definition 2.5.5 When looking at the composite function $f(g(x))$, we have that

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

Using differential notation, if we consider the composite function $(v \circ u)(x)$, we have that

$$\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}.$$

This defined the **chain rule**. ◇

Warning 2.5.6 It is important to consider the input of a function when taking the derivative! In fact, $f'(g(x))$ and $f'(x)$ are different functions... So computing $\frac{dv}{dx}$ gives a different result than computing $\frac{dv}{du}$.

Activity 2.5.7

- (a) Consider the function $f(x) = -x^2 + 5$ and $g(x) = 2x - 1$. Notice that $f(g(x)) = -4x^2 + 4x + 4$. Which of the following is the derivative function of the composite function $f(g(x))$?

A. $-8x + 4$
B. $-4x$
C. $-2x$
D. 2

- (b) One of the options above is a formula for $f'(x) \cdot g'(x)$. Which one? Notice that this is not the same as the derivative of $f(g(x))$!

Activity 2.5.8 Consider the composite function $h(x) = \sqrt{e^x} = e^{\frac{x}{2}}$. For each of the two expressions, find the derivative using the chain rule. Which of the

following expressions are equal to $h'(x)$? Select all!

- | | |
|--|--|
| A. $\frac{1}{2}(e^x)^{\frac{-1}{2}} \cdot e^x$ | D. $e^{\frac{x}{2}} \cdot \frac{1}{2}$ |
| B. $\frac{1}{2}(e^x)^{\frac{3}{2}} \cdot e^x$ | E. $\frac{1}{2}\sqrt{e^x}$ |
| C. $\frac{1}{2}e^{\frac{-x}{2}}$ | F. $\sqrt{e^x} \cdot e^x$ |

Activity 2.5.9 Below you are given the graphs of two functions: $a(x)$ and $b(x)$. Use the graphs to compute values of composite functions and of their derivatives, when possible (there are points where the derivative of these functions is not defined!). Notice that to compute the derivative at a point, you first want to find the derivative as a function of x and then plug in the input you want to study.

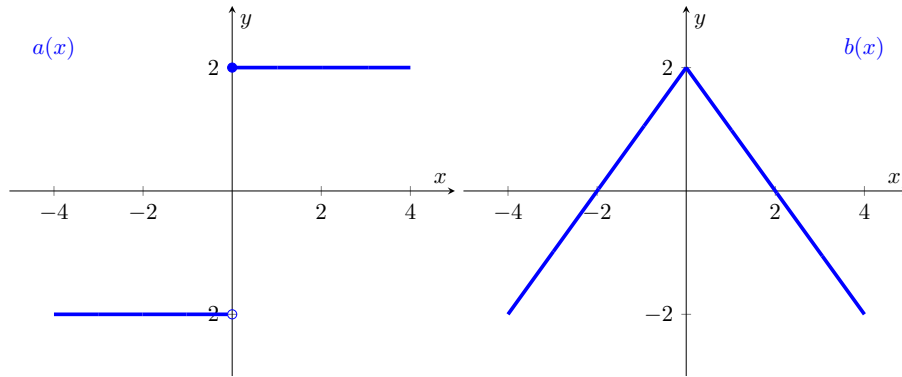


Figure 47 The graphs of $a(x)$ and $b(x)$

- (a) Notice that the derivative of $a \circ b$ is given by $a'(b(x)) \cdot b'(x)$, so the derivative of $a \circ b$ at $x = 4$ is given by the quantity $a'(b(4)) \cdot b'(4) = a'(-2) \cdot b'(4)$, because $b(4) = -2$. Using the graphs to compute slopes, what is the derivative of $a \circ b$ at $x = 4$?
- | | |
|-------|---|
| A. 0 | E. 2 |
| B. -1 | |
| C. 1 | F. The derivative does not exist at this point. |
| D. -2 | |
- (b) Which of the following values is the derivative of $a \circ b$ at $x = 2$?
- | | |
|-------|---|
| A. 0 | E. 2 |
| B. -1 | |
| C. 1 | F. The derivative does not exist at this point. |
| D. -2 | |
- (c) Which of the following values is the derivative of $b \circ a$ (different order!) at $x = -2$?
- | | |
|-------|---|
| A. 0 | E. 2 |
| B. -1 | |
| C. 1 | F. The derivative does not exist at this point. |
| D. -2 | |

Activity 2.5.10 In this activity you will study the derivative of $\cos^n(x)$ for different powers n .

(a) Consider the function $\cos^2(x) = (\cos(x))^2$. Combining power and chain rule, what do you get if you differentiate $\cos^2(x)$?

- A. $-\cos^2(x) \sin(x)$ C. $2 \cos(x) \sin(x)$
 B. $-\cos^2(x) \sin(x)$ D. $-2 \cos(x) \sin(x)$

(b) Consider the function $\cos^3(x)$. Find its derivative.

(c) Consider the function $\cos^n(x)$, for n any number. Find the general formula for its derivative.

Activity 2.5.11 In this activity you will study the derivative of $b^{\cos(x)}$ for different bases b .

(a) Consider the function $e^{\cos(x)}$. Combining exponential and chain rule, what do you get if you differentiate $e^{\cos(x)}$?

- A. $e^{\cos(x)}$ C. $e^{-\sin(x)}$
 B. $-e^{\cos(x)} \sin(x)$ D. $e^{\cos(x)} \sin(x)$

(b) Consider the function $2^{\cos(x)}$. Find its derivative.

(c) Consider the function $b^{\cos(x)}$, for b any positive number. Find the general formula for its derivative.

Remark 2.5.12 Remember that exponential and power functions obey very different differentiation rules. This behavior continues when we consider composite function. The composite power function $f(x)^3$ has derivative

$$3[f(x)]^2 \cdot f'(x)$$

but the composite exponential function $3^{f(x)}$ has derivative

$$\ln(3) 3^{f(x)} \cdot f'(x)$$

Activity 2.5.13 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (chain, product, quotient, sum/difference, etc.) you are using in your work.

1.

$$f(x) = -(4x - 3e^x + 4)^3$$

2.

$$k(w) = 9 \cos\left(w^{\frac{7}{5}}\right)$$

3.

$$h(y) = -3 \sin(-5y^2 + 2y - 5)$$

4.

$$g(t) = 9 \cos(t)^{\frac{7}{5}}$$

Answer.

1.

$$f'(x) = 3(4x - 3e^x + 4)^2(3e^x - 4)$$

2.

$$k'(w) = -\frac{63}{5} w^{\frac{2}{5}} \sin\left(w^{\frac{7}{5}}\right)$$

3.

$$h'(y) = 6(5y - 1) \cos(-5y^2 + 2y - 5)$$

4.

$$g'(t) = -\frac{63}{5} \cos(t)^{\frac{2}{5}} \sin(t)$$

Activity 2.5.14 Notice that

$$\left(\frac{f(x)}{g(x)}\right)' = (f(x) \cdot g(x)^{-1})'$$

Use this observation, the chain rule, the product rule, and the power rule (plus some fraction algebra) to deduce the quotient rule in a new way!

Activity 2.5.15 Remember my neighborhood squirrel invasion? The squirrel population grows based on acorn availability, at a rate of 2 squirrels per bushel of acorns. Acorn availability grows at a rate of 100 bushels of acorns per week. Considering this information as pertaining to the moment $t = 0$, you are given the following possible model for the squirrel:

$$s(a(t)) = 2a(t) + 10 = 2(50 \sin(2t) + 60) + 10.$$

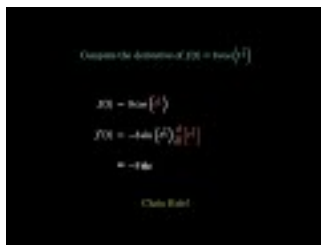
- Check that the model satisfies the data $\frac{ds}{da} = 2$ and $\frac{da}{dt}|_{t=0} = 100$
- Find the derivative function $\frac{ds}{dt}$ and check that $\frac{ds}{dt}|_{t=0} = 200$.
- According to this model, what is the maximum and minimum squirrel population? What is the fastest rate of increase and decrease of the squirrel population? When will these extremal scenarios occur?

Activity 2.5.16 Suppose that a fish population at t months is approximated by

$$P(t) = 100 \cdot 4^{0.05t}$$

- Find $P(10)$ and use units to explain what this value tells us about the population.
- Find $P'(10)$ and use units to explain what this value tells us about the population. (If you want to avoid using a calculator, you can use the approximation $\ln(4) = 1.4$.)

2.5.2 Videos



YouTube: <https://www.youtube.com/watch?v=qGK9bgyTQH8>

Figure 48 Video for DF5

2.6 Differentiation strategy (DF6)

Learning Outcomes

- Compute derivatives using a combination of algebraic derivative rules.

2.6.1 Activities

Activity 2.6.1 Consider the functions defined below:

$$f(x) = \sin((x^2 + 3x) \cos(2x))$$

$$g(x) = \sin(x^2 + 3x) \cos(2x)$$

- What do you notice that is similar about these two functions?
- What do you notice that is different about these two functions?
- Imagine that you are sorting functions into different categories based on how you would differentiate them. In what category (or categories) might these functions fall?

Remark 2.6.2 To take a derivative, we need to examine how the function is built and then proceed accordingly. Below are some questions you might ask yourself as you take the derivative of a function, especially one where multiple rules might need to be used:

- How is this function built algebraically? What kind of function is this? What is the big picture?
- Where do you start?
- Is there an easier or more convenient way to write the function?
- Are there products or quotients involved?
- Is this function a composition of two (or more) elementary functions? If so, what are the outside and inside functions?
- What derivative rules will be needed along the way?

Activity 2.6.3 Consider the function $f(x) = x^3\sqrt{3 - 8x^2}$.

- You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

A. Chain rule	D. Quotient rule
B. Power rule	
C. Product rule	E. Sum/difference rule
- What other rules would be needed along the way? Select all that apply.

A. Chain rule	D. Quotient rule
B. Power rule	
C. Product rule	E. Sum/difference rule
- Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.4 Consider the function $f(x) = \left(\frac{\ln x}{(3x-4)^3}\right)^5$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(b) What other rules would be needed along the way? Select all that apply.

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.5 Consider the function $f(x) = \sin(\cos(\tan(2x^3 - 1)))$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(b) What other rules would be needed along the way? Select all that apply.

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.6 Consider the function $f(x) = \frac{x^2 e^x}{2x^3 - 5x + \sqrt{x}}$.

(a) You will need multiple derivative rules to find $f'(x)$. Which rule would need to be applied first? In other words, what is the big picture here?

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(b) What other rules would be needed along the way? Select all that apply.

- | | |
|-----------------|------------------------|
| A. Chain rule | D. Quotient rule |
| B. Power rule | |
| C. Product rule | E. Sum/difference rule |

(c) Write an outline of the steps needed if you were asked to take the derivative of $f(x)$.

Activity 2.6.7 Find the derivative of the following functions. For each, include an explanation of the steps involved that references the algebraic structure of the function.

(a) $f(x) = e^{5x}(x^2 + 7^x)^3$

(b) $f(x) = \left(\frac{3x+1}{2x^6-1}\right)^5$

(c) $f(x) = \sqrt{\cos(2x^2 + x)}$

(d) $f(x) = \tan(xe^x)$

Activity 2.6.8 Demonstrate and explain how to find the derivative of the following functions. Be sure to explicitly denote which derivative rules (constant multiple, sum/difference, etc.) you are using in your work.

(a)

$$f(y) = \sqrt{\cos(6y^4 - 6y)}$$

(b)

$$g(t) = \left(\frac{5t^3 + 2}{4t^4 - 3}\right)^4$$

(c)

$$h(x) = -(5x^4 - 7x^3)^5 x^{\frac{1}{4}}$$

2.6.2 Videos



YouTube: <https://www.youtube.com/watch?v=kuf14dx9s-A>

Figure 49 Video for DF6

2.7 Differentiating implicitly defined functions (DF7)

Learning Outcomes

- Compute derivatives of implicitly-defined functions.

2.7.1 Activities

Observation 2.7.1 Many of the equations that has been discussed so far fall under the category of an explicit equation. An explicit equation is one in which the relationship between x and y is given explicitly, such as $y = f(x)$. In this section we will examine when the relationship between x and y is given implicitly. An implicit equation looks like $f(x, y) = g(x, y)$ where both sides of the equation may depend on both x and y .

**4.5 solving quadratic equations by factoring (Modeling,
Functions, and Graphs)**

5. Check that your answer to part (4) corresponds to a point on your graph. Approximate from your graph another time at which the baseball is at the same height as your answer to part (4).
6. Use your graph to find two times when the baseball is at a height of 64 feet.
7. Use your graph to approximate two times when the baseball is at a height of 20 feet. Then use the formula to find the actual heights at those times.
8. Suppose the catcher catches the baseball at a height of 4 feet, before it strikes the ground. At what time was the ball caught?
9. Use your calculator to make a table of values for the equation $h = -16t^2 + 64t + 4$ with $TblStart = 0$ and $\Delta Tbl = 0.5$.
10. Use your calculator to graph the equation for the height of the ball, with window settings

$$\begin{aligned} Xmin &= 0, & Xmax &= 4.5, & Yscl &= 5 \\ Ymin &= 0, & Ymax &= 70, & Yscl &= 5 \end{aligned}$$

11. Use the *intersect* command to verify your answer part (7): Estimate two times when the baseball is at a height of 20 feet.
12. Use the *intersect* command to verify your answer to part (8): At what time was the ball caught if it was caught at a height of 4 feet?

Factors and x -Intercepts

In Investigation 6.1, p. 614, perhaps you recognized the graph of the baseball's height as a parabola. In this chapter, we shall see that the graph of any quadratic function is a parabola.

Quadratic Function.

A **quadratic function** is one that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a , b , and c are constants, and a is not equal to zero.

Note 6.1 In the definition above, notice that if a is zero, there is no x -squared term, so the function is not quadratic.

In Investigation 6.1, p. 614, the height of a baseball t seconds after being hit was given by

$$h = -16t^2 + 64t + 4$$

We used a graph to find two times when the baseball was 64 feet high. Can we solve the same problem algebraically?

We are looking for values of t that produce $h = 64$ in the height equation. So, if we substitute $h = 64$ into the height equation, we would like to solve the **quadratic equation**

$$64 = -16t^2 + 64t + 4$$

This equation cannot be solved by extraction of roots, because there are two terms containing the variable t , and they cannot be combined. To solve this

equation, we will appeal to a property of our number system, called the **zero-factor principle**.

Zero-Factor Principle

Can you multiply two numbers together and obtain a product of zero? Only if one of the two numbers happens to be zero. This property of numbers is called the **zero-factor principle**.

Zero-Factor Principle.

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

The principle is true even if the numbers a and b are represented by algebraic expressions, such as $x - 5$ or $2x + 1$. For example, if

$$(x - 5)(2x + 1) = 0$$

then it must be true that either $x - 5 = 0$ or $2x + 1 = 0$. Thus, we can use the zero-factor principle to solve equations.

Example 6.2

- a Solve the equation $(x - 6)(x + 2) = 0$.
- b Find the x -intercepts of the graph of $f(x) = x^2 - 4x - 12$.

Solution.

- a We apply the zero-factor principle to the product $(x - 6)(x + 2)$.

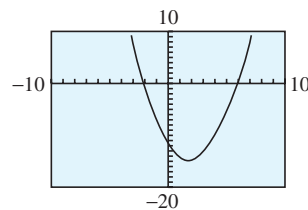
$$\begin{aligned} (x - 6)(x + 2) &= 0 && \text{Set each factor equal to zero.} \\ x - 6 = 0 \quad \text{or} \quad x + 2 = 0 &&& \text{Solve each equation.} \\ x = 6 \quad \text{or} \quad x = -2 &&& \end{aligned}$$

There are two solutions, 6 and -2 . (You should check that both of these values satisfy the original equation.)

- b To find the x -intercepts of the graph, we set $y = 0$ and solve the equation

$$0 = x^2 - 4x - 12$$

But this is the equation we solved in part (a), because $(x - 6)(x + 2) = x^2 - 4x - 12$. The solutions of that equation were 6 and -2 , so the x -intercepts of the graph are 6 and -2 . You can see this by graphing the equation on your calculator, as shown below.



Example 6.2, p. 616 illustrates an important fact about the x -intercepts of a graph.

x -Intercepts of a Graph.

The x -intercepts of the graph of $y = f(x)$ are the solutions of the equation $f(x) = 0$.

Checkpoint 6.3 QuickCheck 1. What are the x -intercepts of the graph of $y = 3(2x - 7)(x + 2)$?

- $\frac{7}{2}$ and -2
- $-\frac{7}{2}$ and 2
- $3, \frac{7}{2}$ and -2
- $3, -\frac{7}{2}$ and 2

Answer. Choice 1

Solution. $\frac{7}{2}$ and -2

Checkpoint 6.4 Practice 1. Graph the function

$$f(x) = (x - 3)(2x + 3)$$

on a calculator, and use your graph to solve the equation $f(x) = 0$. (Use Xmin = -9.4 , Xmax = 9.4 .)

Solutions: $x =$ _____ [Separate different values with a comma.]

Check your answer with the zero-factor principle.

Answer. $-\frac{3}{2}, 3$

Solution. $x = -\frac{3}{2}, x = 3$

Checkpoint 6.5 Pause and Reflect. How can you use a graph to factor a quadratic expression?

Solving Quadratic Equations by Factoring

Before we apply the zero-factor principle to solve a quadratic equation, we must first write the equation so that one side of the equation is zero. Let us introduce some terminology.

Forms for Quadratic Equations.

1. A quadratic equation written

$$ax^2 + bx + c = 0$$

is in **standard form**.

2. A quadratic equation written

$$a(x - r_1)(x - r_2) = 0$$

is in **factored form**.

Once we have written the equation in standard form, we factor the left side and set each variable factor equal to zero separately.

Example 6.6

Solve $3x(x + 1) = 2x + 2$

Solution. First, we write the equation in standard form.

$$3x(x + 1) = 2x + 2 \quad \text{Apply the distributive law to the left side.}$$

$$3x^2 + 3x = 2x + 2 \quad \text{Subtract } 2x + 2 \text{ from both sides.}$$

$$3x^2 + x - 2 = 0$$

Next, we factor the left side to obtain

$$(3x - 2)(x + 1) = 0$$

We then apply the zero-factor principle by setting each factor equal to zero.

$$3x - 2 = 0 \quad \text{or} \quad x + 1 = 0$$

Finally, we solve each equation to find

$$x = \frac{2}{3} \quad \text{or} \quad x = -1$$

The solutions are $\frac{2}{3}$ and -1 .

Caution 6.7 When we apply the zero-factor principle, one side of the equation *must be zero*. For example, to solve the equation

$$(x - 2)(x - 4) = 15$$

it is incorrect to set each factor equal to 15! (There are many ways that the product of two numbers can equal 15; it is not necessary that one of the numbers be 15.) We must first simplify the left side and write the equation in standard form. (The correct solutions are 7 and -1 ; make sure you can find these solutions.)

We summarize the factoring method for solving quadratic equations as follows.

To Solve a Quadratic Equation by Factoring.

- 1 Write the equation in standard form.
- 2 Factor the left side of the equation.
- 3 Apply the zero-factor principle: Set each factor equal to zero.
- 4 Solve each equation. There are two solutions (which may be equal).

Checkpoint 6.8 Practice 2. Solve by factoring: $(t - 3)^2 = 3(9 - t)$

Solutions: $t = \underline{\hspace{2cm}}$ [Separate different values with a comma.]

Answer. $-3, 6$

Solution. After rewriting the equation in standard form and then factoring, we find $t = -3, t = 6$.

We can use factoring to solve the equation from Investigation 6.1, p. 614.

Example 6.9

The height, h , of a baseball t seconds after being hit is given by

$$h = -16t^2 + 64t + 4$$

When will the baseball reach a height of 64 feet?

Solution. We substitute 64 for h in the formula, and solve for t .

$$64 = -16t^2 + 64t + 4 \quad \text{Write the equation in standard form.}$$

$$16t^2 - 64t + 60 = 0 \quad \text{Factor 4 from the left side.}$$

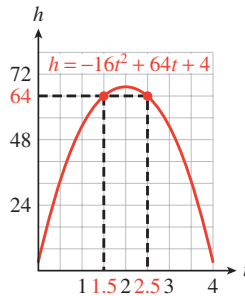
$$4(4t^2 - 16t + 15) = 0 \quad \text{Factor the quadratic expression.}$$

$$4(2t - 3)(2t - 5) = 0 \quad \text{Set each variable factor equal to zero.}$$

$$2t - 3 = 0 \quad \text{or} \quad 2t - 5 = 0 \quad \text{Solve each equation.}$$

$$t = \frac{3}{2} \quad \text{or} \quad t = \frac{5}{2}$$

There are two solutions to the quadratic equation. At $t = \frac{3}{2}$ seconds, the ball reaches a height of 64 feet on the way up, and at $t = \frac{5}{2}$ seconds, the ball is 64 feet high on its way down.



In the solution to Example 6.9, p. 619, the factor 4 does not affect the solutions of the equation at all. You can understand why this is true by looking at some graphs. First, check that the two equations

$$x^2 - 4x + 3 = 0 \quad \text{and} \quad 4(x^2 - 4x + 3) = 0$$

have the same solutions, $x = 1$ and $x = 3$. Then use your graphing calculator to graph the equation

$$Y_1 = X^2 - 4X + 3$$

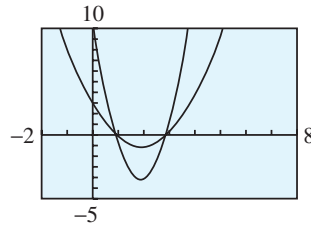
in the window

$$\begin{aligned} X_{\min} &= -2 & X_{\max} &= 8 \\ Y_{\min} &= -5 & Y_{\max} &= 10 \end{aligned}$$

Notice that when $y = 0$, $x = 3$ or $x = 1$. These two points are the x -intercepts of the graph. In the same window, now graph

$$Y_2 = 4(X^2 - 4X + 3)$$

This graph has the same x -values when $y = 0$. The factor of 4 stretches the graph vertically but does not change the location of the x -intercepts.



Checkpoint 6.10 QuickCheck 2. What happens to the x -intercepts when you multiply the right side of $y = ax^2 + bx + c$ by 3?

- They are tripled.
- They are divided by 3.
- They move 3 units to the right.
- They are unchanged.

Answer. They are unchanged.

Solution. They are unchanged.

The value of the constant factor a in the factored form of a quadratic function, $f(x) = a(x - r_1)(x - r_2)$, does not affect the location of the x -intercepts, because it does not affect the solutions of the equation $a(x - r_1)(x - r_2) = 0$.

Checkpoint 6.11 Practice 3.

- a. Solve $f(t) = 4t - t^2 = 0$ by factoring.
Solutions: $t = \underline{\hspace{2cm}}$ [Separate different values with a comma.]
- b. Solve $g(t) = 20t - 5t^2 = 0$ by factoring.
Solutions: $t = \underline{\hspace{2cm}}$ [Separate different values with a comma.]
- c. Graph $y = f(t)$ and $y = g(t)$ together in the window

$$X_{\min} = -2 \quad X_{\max} = 6$$

$$Y_{\min} = -20 \quad Y_{\max} = 25$$

and locate the horizontal intercepts of each graph.

Horizontal intercepts: $\underline{\hspace{2cm}}$ [Separate different ordered pairs with a comma.]

Answer 1. 0, 4

Answer 2. 0, 4

Answer 3. (0, 0), (4, 0)

Solution.

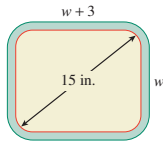
- a. $t = 0, t = 4$
- b. $t = 0, t = 4$
- c. (0, 0), (4, 0)

Checkpoint 6.12 Pause and Reflect. Explain why the solutions of $(x - 3)(x - 6) = 10$ are not 3 and 6.

Applications

Here is another example of how quadratic equations arise in applications.

Example 6.13



The size of a rectangular computer monitor screen is given by the length of its diagonal, as shown at left. If the length of the screen should be 3 inches greater than its width, what are the dimensions of a 15-inch monitor?

Solution. We express the two dimensions of the screen in terms of a single variable:

Width of screen: w

Length of screen: $w + 3$

We can use the Pythagorean theorem to write an equation.

$$w^2 + (w + 3)^2 = 15^2$$

Solve the equation. Begin by simplifying the left side.

$$w^2 + w^2 + 6w + 9 = 225 \quad \text{Write the equation in standard form.}$$

$$2w^2 + 6w - 216 = 0 \quad \text{Factor 2 from the left side.}$$

$$2(w^2 + 3w - 108) = 0 \quad \text{Factor the quadratic expression.}$$

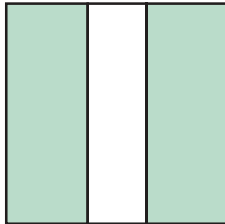
$$2(w - 9)(w + 12) = 0 \quad \text{Set each variable factor equal to zero.}$$

$$w - 9 = 0 \quad \text{or} \quad w + 12 = 0 \quad \text{Solve each equation.}$$

$$w = 9 \quad \text{or} \quad w = -12$$

Because the width of the screen cannot be a negative number, we discard the solution $w = -12$. Thus, the width is $w = 9$ inches, and the length is $w + 3 = 12$ inches.

Checkpoint 6.14 Practice 4.



Francine is designing the layout for a botanical garden. The plan includes a square herb garden, with a path 5 feet wide through the center of the garden, as shown above. To include all the species of herbs, the planted area must be 300 square feet. Find the dimensions of the herb garden.

Answer: $\underline{\quad}$ feet by $\underline{\quad}$ feet

Answer 1. 20

Answer 2. 20

Solution. 20 feet by 20 feet

Checkpoint 6.15 QuickCheck 3. If the perimeter of a rectangle is 56 inches and its width is x inches, what is an expression for its length?

- ⊙ $56 - x$
- ⊙ $28 - x$
- ⊙ $56x$
- ⊙ $\frac{28}{x}$

Answer. Choice 2

Solution. $28 - x$

Solutions of Quadratic Equations

We have seen that the solutions of the quadratic equation

$$a(x - r_1)(x - r_2) = 0$$

are r_1 and r_2 . Thus, if we know the two solutions of a quadratic equation, we can work backward and reconstruct the equation, starting from its factored form. We can then write the equation in standard form by multiplying together the factors.

Example 6.16

Find a quadratic equation whose solutions are $\frac{1}{2}$ and -3 .

Solution. The quadratic equation is

$$\begin{aligned} \left(x - \frac{1}{2}\right) [x - (-3)] &= 0 \\ \left(x - \frac{1}{2}\right) (x + 3) &= 0 \end{aligned}$$

To write the equation in standard form, we multiply the factors together.

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

We can also find an equation with integer coefficients if we clear the equation of fractions. Multiply both sides by 2:

$$\begin{aligned} 2 \left(x^2 + \frac{5}{2}x - \frac{3}{2}\right) &= 2(0) \\ 2x^2 + 5x - 3 &= 0 \end{aligned}$$

You can check that the solutions of this last equation are in fact $\frac{1}{2}$ and -3 . Multiplying both sides of an equation by a constant factor does not change its solutions.

Checkpoint 6.17 QuickCheck 4. Which statement is true?

- ⊙ All rectangles with the same perimeter have the same area.
- ⊙ The solutions of $x(18 - x) = 80$ are 18 and 80.
- ⊙ If the perimeter of a rectangle is 20 cm, the largest area it can have is 20 sq cm.
- ⊙ If you know the x -intercepts of the graph of $y = x^2 + bx + c$, you can

write it in factored form.

Answer. Choice 4

Solution. If you know the x -intercepts of the graph of $y = x^2 + bx + c$, you can write it in factored form.

Checkpoint 6.18 Practice 5. Find a quadratic equation with integer coefficients whose solutions are $\frac{2}{3}$ and -5 . The coefficient of x^2 should be the smallest positive integer coefficient that will work
 $\underline{\hspace{2cm}} = 0$

Answer. $3x^2 + 13x - 10$

Solution. $3x^2 + 13x - 10 = 0$

Note 6.19 A quadratic equation in one variable always has two solutions. However, in some cases, the solutions may be equal. For example, the equation $x^2 - 2x + 1 = 0$ can be solved by factoring as follows:

$$(x - 1)(x - 1) = 0 \quad \text{Apply the zero-factor principle.}$$

$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Both of these equations have solution 1. We say that 1 is a solution of **multiplicity** two, meaning that it occurs twice as a solution of the quadratic equation.

Equations Quadratic in Form

The equation

$$x^6 - 4x^3 - 5 = 0$$

is not quadratic, but if we make the substitution $u = x^3$, the equation becomes

$$u^2 - 4u - 5 = 0$$

An equation is called **quadratic in form** if we can use a substitution to write it as

$$au^2 + bu + c = 0$$

where u stands for an algebraic expression. Such equations can be solved by the same techniques we use to solve quadratic equations.

Example 6.20

Use the substitution $u = x^3$ to solve the equation

$$x^6 - 4x^3 - 5 = 0$$

Solution. We set $u = x^3$, so that $u^2 = (x^3)^2 = x^6$. The original equation then becomes a quadratic equation in the variable u , which we can solve by factoring.

$$u^2 - 4u - 5 = 0$$

Factor the left side.

$$(u + 1)(u - 5) = 0$$

Apply the zero-factor principle.

$$u + 1 = 0 \quad \text{or} \quad u - 5 = 0$$

Solve each equation for u .

$$u = -1 \quad \text{or} \quad u = 5$$

Finally, we replace u by x^3 and solve for x .

$$\begin{aligned} x^3 &= -1 & \text{or} & & x^3 &= 5 & \text{Take cube roots.} \\ x &= \sqrt[3]{-1} = -1 & \text{or} & & x &= \sqrt[3]{5} \end{aligned}$$

You can verify that the solutions of the original equation are -1 and $\sqrt[3]{5}$.

We say that the equation in Example 6.20, p. 623, $x^6 - 4x^3 - 5 = 0$, is **quadratic in x^3** . We chose the substitution $u = x^3$ because $x^6 = u^2$.

Checkpoint 6.21 Practice 6. Use the substitution $u = x^2$ to solve the equation $x^4 - 5x^2 + 6 = 0$.

Solutions: $x =$ _____

Answer. $\sqrt{2}, -(\sqrt{2}), \sqrt{3}, -(\sqrt{3})$

Solution. $x = \pm\sqrt{2}, x = \pm\sqrt{3}$

Usually, you can choose the simpler variable term in the equation for the u -substitution. For example, in Checkpoint 6.21, p. 624 we chose $u = x^2$ because $u^2 = (x^2)^2 = x^4$, which is the first term of the equation. Once you have chosen the u -substitution, you should check that the other variable term is then a multiple of u^2 ; otherwise, the equation is not quadratic in form.

Example 6.22

Solve the equation $e^{2x} - 7e^x + 12 = 0$.

Solution. We use the substitution $u = e^x$, because $u^2 = (e^x)^2 = e^{2x}$. The original equation then becomes

$$\begin{aligned} u^2 - 7u + 12 &= 0 & \text{Factor the left side.} \\ (u - 3)(u - 4) &= 0 & \text{Apply the zero-factor principle.} \\ u - 3 = 0 & \text{ or } & u - 4 = 0 & \text{ Solve each equation for } u. \\ u = 3 & \text{ or } & u = 4 \end{aligned}$$

Finally, we replace u by e^x and solve for x .

$$\begin{aligned} e^x &= 3 & \text{or} & & e^x &= 4 \\ x &= \ln 3 & \text{or} & & x &= \ln 4 \end{aligned}$$

You should verify that the solutions of the original equation are $\ln 3$ and $\ln 4$.

Checkpoint 6.23 Practice 7. Solve the equation $10^{2x} - 3 \cdot 10^x + 2 = 0$, and check the solutions.

Solutions: $x =$ _____

Answer. $0, \log(2)$

Solution. $x = 0, x = \log 2$

Checkpoint 6.24 Pause and Reflect. Explain why we cannot “cancel” $(x - 5)$ from both sides of the equation $3x(x - 5) = 6(x - 5)$. What are the solutions of the equation?

Section Summary

Vocabulary

Look up the definitions of new terms in the Glossary.

- Quadratic function
- Zero-factor principle
- Standard form
- Factored form
- Multiplicity
- Monotonic

CONCEPTS

Quadratic Function.

- 1 A **quadratic function** is one that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a , b , and c are constants, and a is not equal to zero.

Zero-Factor Principle.

- 2 The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

x -Intercepts of a Graph.

- 3 The x -intercepts of the graph of $y = f(x)$ are the solutions of the equation $f(x) = 0$.

- 4 A quadratic equation written as $ax^2 + bx + c = 0$ is in **standard form**. A quadratic equation written as $a(x - r_1)(x - r_2) = 0$ is in **factored form**.

To Solve a Quadratic Equation by Factoring.

5

- 1 Write the equation in standard form.
- 2 Factor the left side of the equation.
- 3 Apply the zero-factor principle: Set each factor equal to zero.
- 4 Solve each equation. There are two solutions (which may be equal).

- 6 Every quadratic equation has two solutions, which may be the same.
- 7 The value of the constant a in the factored form of a quadratic equation does not affect the solutions.

- 8 Each solution of a quadratic equation corresponds to a factor in the factored form.
- 9 An equation is called **quadratic in form** if we can use a substitution to write it as $au^2 + bu + c = 0$, where u stands for an algebraic expression.

STUDY QUESTIONS

- 1
 - a Find a pair of numbers whose product is 6. Now find a different pair of numbers whose product is 6. Can you find more such pairs?
 - b Find a pair of numbers whose product is 0. What is true about any such pair?
- 2 Before you begin factoring to solve a quadratic equation, what should you do?
- 3 How can you find the x -intercepts of the graph of $y = f(x)$ without looking at the graph?
- 4 How many solutions does a quadratic equation have?
- 5
 - a Write a linear equation whose only solution is $x = 3$.
 - b Write a quadratic equation whose only solution is $x = 3$.
- 6 If you know the solutions of $ax^2 + bx + c = 0$, how can you find the solutions of $5(ax^2 + bx + c) = 0$?
- 7 Is the equation $x^9 - 6x^3 + 8 = 0$ quadratic in form? Why or why not?
- 8 Delbert says that he can solve the equation $x(x + 5) = 2(x + 5)$ by canceling the factor $(x + 5)$ to get $x = 2$. Comment on his method

SKILLS

Practice each skill in the Homework problems listed.

- 1 Use the zero-factor principle and find x -intercepts: #3–10
- 2 Solve quadratic equations by factoring: #11–24
- 3 Use the x -intercepts of the graph to factor a quadratic equation: #25–28, 37–40
- 4 Write a quadratic equation with given solutions: #29–36
- 5 Solve applied problems involving quadratic equations: #41–50
- 6 Solve equations that are quadratic in form: #51–62

Homework 6.1

1. Delbert stands at the top of a 300-foot cliff and throws his algebra book directly upward with a velocity of 20 feet per second. The height of his book above the ground t seconds later is given by the equation

$$h = -16t^2 + 20t + 300$$

where h is in feet.

- (a) Use your calculator to make a table of values for the height equation, with increments of 0.5 second.

- (b) Graph the height equation on your calculator. Use your table of values to help you choose appropriate window settings.
- (c) What is the highest altitude Delbert's book reaches? When does it reach that height? Use the **TRACE** feature to find approximate answers first. Then use the **Table** feature to improve your estimate.
- (d) When does Delbert's book pass him on its way down? (Delbert is standing at a height of 300 feet.) Use the **intersect** command.
- (e) How long will it take Delbert's book to hit the ground at the bottom of the cliff?
2. James Bond stands on top of a 240-foot building and throws a film canister upward to a fellow agent in a helicopter 16 feet above the building. The height of the film above the ground t seconds later is given by the formula

$$h = -16t^2 + 32t + 240$$

where h is in feet.

- (a) Use your calculator to make a table of values for the height formula, with increments of 0.5 second.
- (b) Graph the height formula on your calculator. Use your table of values to help you choose appropriate window settings.
- (c) How long will it take the film canister to reach the agent in the helicopter? (What is the agent's altitude?) Use the **TRACE** feature to find approximate answers first. Then use the **Table** feature to improve your estimate.
- (d) If the agent misses the canister, when will it pass James Bond on the way down? Use the **intersect** command.
- (e) How long will it take it to hit the ground?

In Problems 3–10, use a graph to solve the equation $y = 0$. (Use $X_{\min} = -9.4$, $X_{\max} = 9.4$.) Check your answers with the zero-factor principle.

- | | |
|--------------------------|--------------------------|
| 3. $y = (2x + 5)(x - 2)$ | 4. $y = (x + 1)(4x - 1)$ |
| 5. $y = x(3x + 10)$ | 6. $y = x(3x - 7)$ |
| 7. $y = (4x + 3)(x + 8)$ | 8. $y = (x - 2)(x - 9)$ |
| 9. $y = (x - 4)^2$ | 10. $y = (x + 6)^2$ |

For Problems 11–24, solve by factoring. (See Algebra Skills Refresher Appendix A.8, p. 1045 to review factoring.)

- | | |
|-----------------------------|---------------------------------|
| 11. $2a^2 + 5a - 3 = 0$ | 12. $3b^2 - 4b - 4 = 0$ |
| 13. $2x^2 = 6x$ | 14. $5z^2 = 5z$ |
| 15. $3y^2 - 6y = -3$ | 16. $4y^2 + 4y = 8$ |
| 17. $x(2x - 3) = -1$ | 18. $2x(x - 2) = x + 3$ |
| 19. $t(t - 3) = 2(t - 3)$ | 20. $5(t + 2) = t(t + 2)$ |
| 21. $z(3z + 2) = (z + 2)^2$ | 22. $(z - 1)^2 = 2z^2 + 3z - 5$ |
| 23. $(v + 2)(v - 5) = 8$ | 24. $(w + 1)(2w - 3) = 3$ |

In Problems 25–28, graph each set of functions in the standard window. What do you notice about the x -intercepts? Generalize your observation, and test

your idea with examples.

25.

(a) $f(x) = x^2 - x - 20$

(b) $g(x) = 2(x^2 - x - 20)$

(c) $h(x) = 0.5(x^2 - x - 20)$

27.

(a) $f(x) = x^2 + 6x - 16$

(b) $g(x) = -2(x^2 + 6x - 16)$

(c) $h(x) = -0.1(x^2 + 6x - 16)$

26.

(a) $f(x) = x^2 + 2x - 15$

(b) $g(x) = 3(x^2 + 2x - 15)$

(c) $h(x) = 0.2(x^2 + 2x - 15)$

28.

(a) $f(x) = x^2 - 16$

(b) $g(x) = -1.5(x^2 - 16)$

(c) $h(x) = -0.4(x^2 - 16)$

In Problems 29–36, write a quadratic equation whose solutions are given. The equation should be in standard form with integer coefficients.

29. -2 and 1

30. -4 and 3

31. 0 and -5

32. 0 and 5

33. -3 and $\frac{1}{2}$

34. $\frac{-2}{3}$ and 4

35. $\frac{-1}{4}$ and $\frac{3}{2}$

36. $\frac{-1}{3}$ and $\frac{-1}{2}$

For problems 37–40, graph the function in the **ZInteger** window, and locate the x -intercepts of the graph. Use the x -intercepts to write the quadratic expression in factored form.

37. $f(x) = 0.1(x^2 - 3x - 270)$

38. $h(x) = 0.1(x^2 + 9x - 360)$

39. $g(x) = -0.08(x^2 + 14x - 576)$

40. $F(x) = -0.06(x^2 - 22x - 504)$

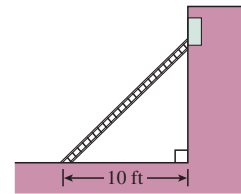
Use the Pythagorean theorem to solve Problems 41 and 42. (See Algebra Skills Refresher Appendix A.11, p. 1074 to review the Pythagorean theorem.)

41.

One end of a ladder is 10 feet from the base of a wall, and the other end reaches a window in the wall. The ladder is 2 feet longer than the height of the window.

(a) Write a quadratic equation about the height of the window.

(b) Solve your equation to find the height of the window.



42. The diagonal of a rectangle is 20 inches. One side of the rectangle is 4 inches shorter than the other side.

(a) Write a quadratic equation about the length of the rectangle.

(b) Solve your equation to find the dimensions of the rectangle.

Use the following formula to answer Problems 43 and 44. If an object is thrown into the air from a height s_0 above the ground with an initial velocity v_0 , its height t seconds later is given by the formula

$$h = -\frac{1}{2}gt^2 + v_0t + s_0$$

where g is a constant that measures the force of gravity.

- 43.** A tennis ball is thrown into the air with an initial velocity of 16 feet per second from a height of 8 feet. The value of g is 32.
- Write a quadratic equation that gives the height of the tennis ball at time t .
 - Find the height of the tennis ball at $t = \frac{1}{2}$ second and at $t = 1$ second.
 - Write and solve an equation to answer the question: At what time is the tennis ball 11 feet high?
 - Use the Table feature on your calculator to verify your answers to parts (b) and (c). (What value of ΔT_{bl} is useful for this problem?)
 - Graph your equation from part (a) on your calculator. Use your table to help you choose an appropriate window.
 - If nobody hits the tennis ball, approximately how long will it be in the air?
- 44.** A mountain climber stands on a ledge 80 feet above the ground and tosses a rope down to a companion clinging to the rock face below the ledge. The initial velocity of the rope is -8 feet per second, and the value of g is 32.
- Write a quadratic equation that gives the height of the rope at time t .
 - What is the height of the rope after $\frac{1}{2}$ second? After 1 second?
 - Write and solve an equation to answer the question: How long does it take the rope to reach the second climber, who is 17 feet above the ground?
 - Use the Table feature on your calculator to verify your answers to parts (b) and (c). (What value of ΔT_{bl} is useful for this problem?)
 - Graph your equation from part (a) on your calculator. Use your table to help you choose an appropriate window.
 - If the second climber misses the rope, approximately how long will the rope take to reach the ground?

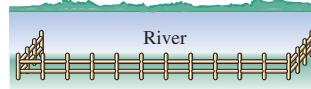
For Problems 45 and 46, you may want to review Investigation 2.2, p. 163, Perimeter and Area, in Chapter 2, p. 161.

- 45.** A rancher has 360 yards of fence to enclose a rectangular pasture. If the pasture should be 8000 square yards in area, what should its dimensions be? We will use 3 methods to solve this problem: a table of values, a graph, and an algebraic equation.
- Make a table by hand that shows the areas of pastures of various widths, as shown here.

Width	Length	Area
10	170	1700
\vdots	\vdots	\vdots

(To find the length of each pasture, ask yourself, What is the sum of the length plus the width if there are 360 yards of fence?) Continue the table until you find the pasture whose area is 8000 square yards.

- (b) Write an expression for the length of the pasture if its width is x . Next, write an expression for the area, A , of the pasture if its width is x . Graph the equation for A on your calculator, and use the graph to find the pasture of area 8000 square yards.
- (c) Write an equation for the area, A , of the pasture in terms of its width x . Solve your equation algebraically for $A = 8000$. Explain why there are two solutions.
- 46.** If the rancher in Problem 45 uses a riverbank to border one side of the pasture as shown in the figure, he can enclose 16,000 square yards with 360 yards of fence. What will the dimensions of the pasture be then? We will use three methods to solve this problem: a table of values, a graph, and an algebraic equation.



- (a) Make a table by hand that shows the areas of pastures of various widths, as shown here.

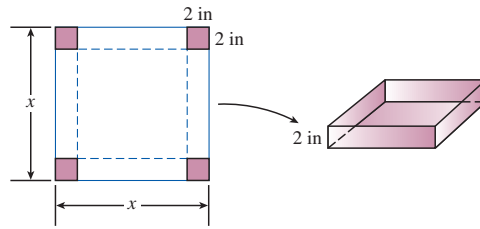
Width	Length	Area
10	340	3400
20	320	6400
\vdots	\vdots	\vdots

(Be careful computing the length of the pasture: Remember that one side of the pasture does not need any fence!) Continue the table until you find the pasture whose area is 16,000 square yards.

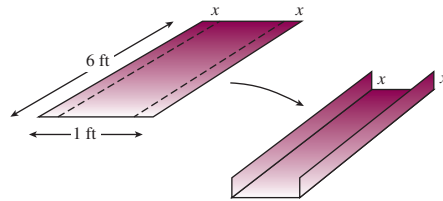
- (b) Write an expression for the length of the pasture if its width is x . Next, write an expression for the area, A , of the pasture if its width is x . Graph the equation for A , and use the graph to find the pasture of area 16,000 square yards.
- (c) Write an equation for the area, A , of the pasture in terms of its width x . Solve your equation algebraically for $A = 16,000$.

For Problems 47 and 48, you will need the formula for the volume of a box.

- 47.** A box is made from a square piece of cardboard by cutting 2-inch squares from each corner and turning up the edges.



- (a) If the piece of cardboard is x inches square, write expressions for the length, width, and height of the box. Then write an expression for the volume, V , of the box in terms of x .
- (b) Use your calculator to make a table of values showing the volumes of boxes made from cardboard squares of side 4 inches, 5 inches, and so on.
- (c) Graph your expression for the volume on your calculator. What happens to V as x increases?
- (d) Use your table or your graph to find what size cardboard you need to make a box with volume 50 cubic inches.
- (e) Write and solve a quadratic equation to answer part (d).
48. A length of rain gutter is made from a piece of aluminum 6 feet long and 1 foot wide.



- (a) If a strip of width x is turned up along each long edge, write expressions for the length, width, and height of the gutter. Then write an expression for the volume, V , of the gutter in terms of x .
- (b) Use your calculator to make a table of values showing the volumes of various rain gutters formed by turning up edges of 0.1 foot, 0.2 foot, and so on.
- (c) Graph your expression for the volume. What happens to V as x increases?
- (d) Use your table or your graph to discover how much metal should be turned up along each long edge so that the gutter has a capacity of $\frac{3}{4}$ cubic foot of rainwater.
- (e) Write and solve a quadratic equation to answer part (d).

Problems 49 and 50 deal with wildlife management. The annual increase, I , in a population often depends on the size x of the population, according to the formula

$$I = kCx - kx^2$$

where k and C are constants related to the fertility of the population and the availability of food.

- 49.** The annual increase, $f(x)$, in the deer population in a national park is given by

$$f(x) = 1.2x - 0.0002x^2$$

where x is the size of the population that year.

- Make a table of values for $f(x)$ for $0 \leq x \leq 7000$. Use increments of 500 in x .
 - How much will a population of 2000 deer increase? A population of 5000 deer? A population of 7000 deer?
 - Use your calculator to graph the annual increase, $f(x)$, versus the size of the population, x , for $0 \leq x \leq 7000$.
 - What do the x -intercepts tell us about the deer population?
 - Estimate the population size that results in the largest annual increase. What is that increase?
- 50.** Commercial fishermen rely on a steady supply of fish in their area. To avoid overfishing, they adjust their harvest to the size of the population. The function

$$g(x) = 0.4x - 0.0001x^2$$

gives the annual rate of growth, in tons per year, of a fish population of biomass x tons.

- Make a table of values for $g(x)$ for $0 \leq x \leq 5000$. Use increments of 500 in x .
- How much will a population of 1000 tons increase? A population of 3000 tons? A population of 5000 tons?
- Use your calculator to graph the annual increase, $g(x)$, versus the size of the population, x , for $0 \leq x \leq 5000$.
- What do the x -intercepts tell us about the fish population?
- Estimate the population size that results in the largest annual increase. What is that increase?

For Problems 51-62, use a substitution to solve the equation.

51. $a^4 + a^2 - 2 = 0$

52. $t^6 - t^3 - 6 = 0$

53. $4b^6 - 3 = b^3$

54. $3x^4 + 1 = 4x^2$

55. $c^{2/3} + 2c^{1/3} - 3 = 0$

56. $y^{1/2} - 3y^{1/4} - 4 = 0$

57. $10^{2w} - 5 \cdot 10^w + 6 = 0$

58. $e^{2x} - 5e^x + 4 = 0$

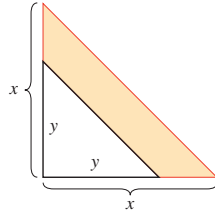
59. $5^{2t} - 30 \cdot 5^t + 125 = 0$

60. $e^{4r} - 3e^{2r} + 2 = 0$

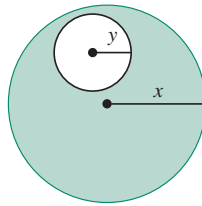
61. $\frac{1}{m^2} + \frac{5}{m} - 6 = 0$

62. $\frac{1}{s^2} + \frac{4}{s} - 5 = 0$

- 63.** The sail in the figure is a right triangle of base and height x . It has a colored stripe along the hypotenuse and a white triangle of base and height y in the lower corner.



- (a) Write an expression for the area of the colored stripe.
- (b) Express the area of the stripe in factored form.
- (c) If the sail is $7\frac{1}{2}$ feet high and the white strip is $4\frac{1}{2}$ feet high, use your answer to (b) to calculate mentally the area of the stripe.
64. An hors d'oeuvres tray has radius x , and the dip container has radius y , as shown in the figure.



- (a) Write an expression for the area for the chips (shaded region).
- (b) Express the area in factored form.
- (c) If the tray has radius $8\frac{1}{2}$ inches and the space for the dip has radius $2\frac{1}{2}$ inches, use your answer to part (b) to calculate mentally the area for chips. (Express your answer as a multiple of π .)

Solving Quadratic Equations

Not every quadratic equation can be solved by factoring or by extraction of roots. For example, the expression $x^2 + x - 1$ cannot be factored, so the equation $x^2 + x - 1 = 0$ cannot be solved by factoring. For other equations, factoring may be difficult. In this section we learn two methods that can be used to solve any quadratic equation.

Squares of Binomials

In Section 2.1, p. 165 we used extraction of roots to solve equations of the form

$$a(px + q)^2 + r = 0$$

where the left side of the equation includes the square of a binomial, or a **perfect square**. We can write any quadratic equation in this form by **completing the square**.

Consider the following squares of binomials.

Square of binomial $(x + p)^2$	p	$2p$	p^2
1. $(x + \mathbf{5})^2 = x^2 + 10x + 25$	$\mathbf{5}$	$2(\mathbf{5}) = 10$	$\mathbf{5}^2 = 25$
2. $(x - \mathbf{3})^2 = x^2 - 6x + 9$	$\mathbf{-3}$	$2(\mathbf{-3}) = -6$	$\mathbf{-3}^2 = 9$
3. $(x - \mathbf{12})^2 = x^2 - 24x + 144$	$\mathbf{-12}$	$2(\mathbf{-12}) = -24$	$\mathbf{-12}^2 = 144$

**4.6 interval notation and solving inequalities (OpenStax
College Algebra with Corequisite Support)**



Figure 1

It is not easy to make the honor roll at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in [Figure 2](#). The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

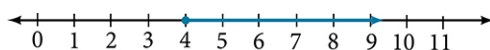


Figure 2

We can use set-builder notation: $\{x|x \geq 4\}$, which translates to “all real numbers x such that x is greater than or equal to 4.” Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be “equaled.” A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6 , including -2 , but not including 6 ; $(-1, 0)$, all real numbers between, but not including -1 and 0 ; and $(-\infty, 1]$, all real numbers less than and including 1 . [Table 1](#) outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b , but not including a or b	$\{x a < x < b\}$	(a, b)
All real numbers greater than a , but not including a	$\{x x > a\}$	(a, ∞)
All real numbers less than b , but not including b	$\{x x < b\}$	$(-\infty, b)$
All real numbers greater than a , including a	$\{x x \geq a\}$	$[a, \infty)$
All real numbers less than b , including b	$\{x x \leq b\}$	$(-\infty, b]$
All real numbers between a and b , including a	$\{x a \leq x < b\}$	$[a, b)$
All real numbers between a and b , including b	$\{x a < x \leq b\}$	$(a, b]$

Table 1

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b , including a and b	$\{x a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Table 1

EXAMPLE 1**Using Interval Notation to Express All Real Numbers Greater Than or Equal to a**

Use interval notation to indicate all real numbers greater than or equal to -2 .

Solution

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

TRY IT #1 Use interval notation to indicate all real numbers between and including -3 and 5 .

EXAMPLE 2**Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b**

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1 .

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1 . So, this interval begins at $-\infty$ and ends at -1 , which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1 , which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, \cup , between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

TRY IT #2 Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

Properties of Inequalities**Addition Property**

If $a < b$, then $a + c < b + c$.

Multiplication Property

If $a < b$ and $c > 0$, then $ac < bc$.

If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

EXAMPLE 3**Demonstrating the Addition Property**

Illustrate the addition property for inequalities by solving each of the following:

1. Ⓐ $x - 15 < 4$
2. Ⓑ $6 \geq x - 1$
3. Ⓒ $x + 7 > 9$

✓ **Solution**

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

<p>Ⓐ</p> $x - 15 < 4$ $x - 15 + 15 < 4 + 15$ $x < 19$	<p>Ⓑ</p> $6 \geq x - 1$ $6 + 1 \geq x - 1 + 1$ $7 \geq x$	<p>Add 15 to both sides.</p> <p>Add 1 to both sides.</p>
<p>Ⓒ</p> $x + 7 > 9$ $x + 7 - 7 > 9 - 7$ $x > 2$	<p>Subtract 7 from both sides.</p>	

> **TRY IT** #3 Solve: $3x - 2 < 1$.

EXAMPLE 4**Demonstrating the Multiplication Property**

Illustrate the multiplication property for inequalities by solving each of the following:

1. Ⓐ $3x < 6$
2. Ⓑ $-2x - 1 \geq 5$
3. Ⓒ $5 - x > 10$

✓ **Solution**

<p>Ⓐ</p> $3x < 6$ $\frac{1}{3}(3x) < (6)\frac{1}{3}$ $x < 2$	<p>Ⓑ</p> $-2x - 1 \geq 5$ $-2x \geq 6$ $\left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right)$ $x \leq -3$	<p>Multiply by $-\frac{1}{2}$.</p> <p>Reverse the inequality.</p>
<p>Ⓒ</p> $5 - x > 10$ $-x > 5$ $(-1)(-x) > (5)(-1)$ $x < -5$	<p>Multiply by -1.</p> <p>Reverse the inequality.</p>	

> **TRY IT** #4 Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

EXAMPLE 5**Solving an Inequality Algebraically**Solve the inequality: $13 - 7x \geq 10x - 4$.✔ **Solution**

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{array}{ll} 13 - 7x \geq 10x - 4 & \\ 13 - 17x \geq -4 & \text{Move variable terms to one side of the inequality.} \\ -17x \geq -17 & \text{Isolate the variable term.} \\ x \leq 1 & \text{Dividing both sides by } -17 \text{ reverses the inequality.} \end{array}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

> **TRY IT** #5 Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

EXAMPLE 6**Solving an Inequality with Fractions**Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.✔ **Solution**

We begin solving in the same way we do when solving an equation.

$$\begin{array}{ll} -\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x & \\ -\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} & \text{Put variable terms on one side.} \\ -\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} & \text{Write fractions with common denominator.} \\ -\frac{17}{12}x \geq -\frac{5}{8} & \\ x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) & \text{Multiplying by a negative number reverses the inequality.} \\ x \leq \frac{15}{34} & \end{array}$$

The solution set is the interval $(-\infty, \frac{15}{34}]$.

> **TRY IT** #6 Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

EXAMPLE 7**Solving a Compound Inequality**Solve the compound inequality: $3 \leq 2x + 2 < 6$.✔ **Solution**The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$\begin{array}{rcl} 3 \leq 2x + 2 & \text{and} & 2x + 2 < 6 \\ 1 \leq 2x & & 2x < 4 \\ \frac{1}{2} \leq x & & x < 2 \end{array}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $[\frac{1}{2}, 2)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$\begin{array}{rcl} 3 \leq 2x + 2 < 6 & & \\ 1 \leq 2x < 4 & \text{Isolate the variable term, and subtract 2 from all three parts.} & \\ \frac{1}{2} \leq x < 2 & \text{Divide through all three parts by 2.} & \end{array}$$

We get the same solution: $[\frac{1}{2}, 2)$.

TRY IT #7 Solve the compound inequality: $4 < 2x - 8 \leq 10$.

EXAMPLE 8

Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution

Let's try the first method. Write two inequalities:

$$\begin{array}{rcl} 3 + x > 7x - 2 & \text{and} & 7x - 2 > 5x - 10 \\ 3 > 6x - 2 & & 2x - 2 > -10 \\ 5 > 6x & & 2x > -8 \\ \frac{5}{6} > x & & x > -4 \\ x < \frac{5}{6} & & -4 < x \end{array}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $(-4, \frac{5}{6})$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See [Figure 3](#).



Figure 3

TRY IT #8 Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x , as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, \quad |A| \leq B, \quad |A| > B, \quad \text{or} \quad |A| \geq B,$$

Where A , and sometimes B , represents an algebraic expression dependent on a variable x . Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of x -values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x - 600|$, and therefore, $|x - 600| \leq 200$ or

$$\begin{aligned} -200 &\leq x - 600 \leq 200 \\ -200 + 600 &\leq x - 600 + 600 \leq 200 + 600 \\ 400 &\leq x \leq 800 \end{aligned}$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Absolute Value Inequalities

For an algebraic expression X , and $k > 0$, an **absolute value inequality** is an inequality of the form

$$\begin{aligned} |X| < k &\text{ is equivalent to } -k < X < k \\ |X| > k &\text{ is equivalent to } X < -k \text{ or } X > k \end{aligned}$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

EXAMPLE 9

Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in [Figure 4](#), to represent the condition to be satisfied.

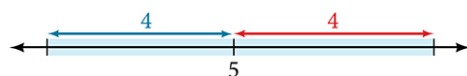


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x - 5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x - 5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{aligned} x - 5 &\leq 4 && \text{and} && x - 5 &\geq -4 \\ x &\leq 9 && && x &\geq 1 \end{aligned}$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x - 5| \leq 4$ is equivalent to $[1, 9]$ in interval notation.

 **TRY IT** #9 Describe all x -values within a distance of 3 from the number 2.

EXAMPLE 10**Solving an Absolute Value Inequality**Solve $|x - 1| \leq 3$.✔ **Solution**

$$|x - 1| \leq 3$$

$$-3 \leq x - 1 \leq 3$$

$$-2 \leq x \leq 4$$

$$[-2, 4]$$

EXAMPLE 11**Using a Graphical Approach to Solve Absolute Value Inequalities**Given the equation $y = -\frac{1}{2}|4x - 5| + 3$, determine the x -values for which the y -values are negative.✔ **Solution**We are trying to determine where $y < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x - 5| < -3 \quad \text{Multiply both sides by } -2, \text{ and reverse the inequality.}$$

$$|4x - 5| > 6$$

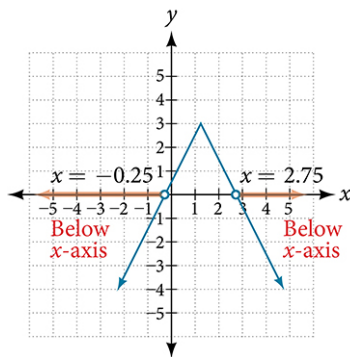
Next, we solve for the equality $|4x - 5| = 6$.

$$4x - 5 = 6 \qquad 4x - 5 = -6$$

$$4x = 11 \qquad \text{or} \qquad 4x = -1$$

$$x = \frac{11}{4} \qquad \qquad \qquad x = -\frac{1}{4}$$

Now, we can examine the graph to observe where the y -values are negative. We observe where the branches are below the x -axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$, and that the graph opens downward. See [Figure 5](#).

**Figure 5**

> **TRY IT** #10 Solve $-2|k - 4| \leq -6$.

▶ **MEDIA**

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

[Interval notation \(http://openstax.org//intervalnotn\)](http://openstax.org//intervalnotn)
[How to solve linear inequalities \(http://openstax.org//solvelinineq\)](http://openstax.org//solvelinineq)
[How to solve an inequality \(http://openstax.org//solveineq\)](http://openstax.org//solveineq)
[Absolute value equations \(http://openstax.org//absvaleq\)](http://openstax.org//absvaleq)
[Compound inequalities \(http://openstax.org//compndineqs\)](http://openstax.org//compndineqs)
[Absolute value inequalities \(http://openstax.org//absvalineqs\)](http://openstax.org//absvalineqs)



2.7 SECTION EXERCISES

Verbal

- When solving an inequality, explain what happened from Step 1 to Step 2:
 Step 1 $-2x > 6$
 Step 2 $x < -3$
- When solving an inequality, we arrive at:
 $x + 2 < x + 3$
 $2 < 3$
 Explain what our solution set is.
- When writing our solution in interval notation, how do we represent all the real numbers?
- When solving an inequality, we arrive at:
 $x + 2 > x + 3$
 $2 > 3$
 Explain what our solution set is.
- Describe how to graph $y = |x - 3|$

Algebraic

For the following exercises, solve the inequality. Write your final answer in interval notation.

- $4x - 7 \leq 9$
- $3x + 2 \geq 7x - 1$
- $-2x + 3 > x - 5$
- $4(x + 3) \geq 2x - 1$
- $-\frac{1}{2}x \leq -\frac{5}{4} + \frac{2}{5}x$
- $-5(x - 1) + 3 > 3x - 4 - 4x$
- $-3(2x + 1) > -2(x + 4)$
- $\frac{x+3}{8} - \frac{x+5}{5} \geq \frac{3}{10}$
- $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

- $|x + 9| \geq -6$
- $|2x + 3| < 7$
- $|3x - 1| > 11$
- $|2x + 1| + 1 \leq 6$
- $|x - 2| + 4 \geq 10$
- $|-2x + 7| \leq 13$
- $|x - 7| < -4$
- $|x - 20| > -1$
- $\left| \frac{x-3}{4} \right| < 2$

For the following exercises, describe all the x -values within or including a distance of the given values.

- Distance of 5 units from the number 7
- Distance of 3 units from the number 9
- Distance of 10 units from the number 4

27. Distance of 11 units from the number 1

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

28. $-4 < 3x + 2 \leq 18$ 29. $3x + 1 > 2x - 5 > x - 7$ 30. $3y < 5 - 2y < 7 + y$
31. $2x - 5 < -11$ or $5x + 1 \geq 6$ 32. $x + 7 < x + 2$

Graphical

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

33. $|x - 1| > 2$ 34. $|x + 3| \geq 5$ 35. $|x + 7| \leq 4$
36. $|x - 2| < 7$ 37. $|x - 2| < 0$

For the following exercises, graph both straight lines (left-hand side being y_1 and right-hand side being y_2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines.

38. $x + 3 < 3x - 4$ 39. $x - 2 > 2x + 1$ 40. $x + 1 > x + 4$
41. $\frac{1}{2}x + 1 > \frac{1}{2}x - 5$ 42. $4x + 1 < \frac{1}{2}x + 3$

Numeric


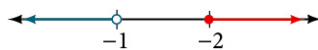

For the following exercises, write the set in interval notation.

43. $\{x | -1 < x < 3\}$ 44. $\{x | x \geq 7\}$ 45. $\{x | x < 4\}$
46. $\{x | x \text{ is all real numbers}\}$

For the following exercises, write the interval in set-builder notation.

47. $(-\infty, 6)$ 48. $(4, \infty)$ 49. $[-3, 5)$
50. $[-4, 1] \cup [9, \infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

51.  52.  53. 

Technology

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter $y_2 =$ the right-hand side. Entering the absolute value of an expression is found in the MATH menu, Num, 1:abs(. Find the points of intersection, recall (2nd CALC 5:intersection, 1st curve, enter, 2nd curve, enter, guess, enter). Copy a sketch of the graph and shade the x-axis for your solution set to the inequality. Write final answers in interval notation.

54. $|x + 2| - 5 < 2$

55. $\frac{-1}{2}|x + 2| < 4$

56. $|4x + 1| - 3 > 2$

57. $|x - 4| < 3$

58. $|x + 2| \geq 5$

Extensions

59. Solve $|3x + 1| = |2x + 3|$

60. Solve $x^2 - x > 12$

61. $\frac{x-5}{x+7} \leq 0, x \neq -7$

62. $p = -x^2 + 130x - 3000$ is a profit formula for a small business. Find the set of x -values that will keep this profit positive.

Real-World Applications

63. In chemistry the volume for a certain gas is given by $V = 20T$, where V is measured in cc and T is temperature in °C. If the temperature varies between 80°C and 120°C, find the set of volume values.
64. A basic cellular package costs \$20/mo. for 60 min of calling, with an additional charge of \$.30/min beyond that time.. The cost formula would be $C = 20 + .30(x - 60)$. If you have to keep your bill no greater than \$50, what is the maximum calling minutes you can use?

5 Topics for Applications of Derivatives

5.1 modeling with functions

5.1.1 modeling with functions (OpenStax College Algebra with Corequisite Support)

5.1.2 functions and modeling (Active Prelude to Calculus)

5.1.3 activities for applied optimization (TBIL)

5.2 circles, traversing a circle, circular functions (Active Prelude to Calculus)

5.3 right triangles

5.3.1 right triangles/ratios (Active Prelude to Calculus)

5.3.2 using inverse trig functions to find angles in triangles (Active Prelude to Calculus)

5.3.3 defining trig functions with right triangles, special right triangles, using trig functions to find side length and in applications (OpenStax Algebra and Trigonometry)

5.3.4 defining trig functions with unit circle (OpenStax Algebra and Trigonometry)

5.4 graphs of sine and cosine functions (OpenStax Algebra and Trigonometry)

Calculus Fun Fact: Economists call calculus their international language. They rely on it to examine functional relationships.

5.1 modeling with functions

5.1.1 modeling with functions (OpenStax College Algebra with Corequisite Support)

5.1.2 functions and modeling (Active Prelude to Calculus)

5.1.3 activities for applied optimization (TBIL)

2.3 Models and Applications

Learning Objectives

In this section, you will:

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

COREQUISITE SKILLS

Learning Objectives

- Solve a formula for a specified variable (IA 2.3.1)
- Use a problem-solving strategy for word problems (IA 2.2.1)

Objective 1: Solve a formula for a specified variable (IA 2.3.1)



HOW TO

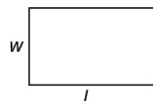
Solving a Formula for a Specified Variable.

- Step 1. Refer to the appropriate formula and identify the variable you are solving for. Treat the other variable terms as if they were numbers.
- Step 2. Bring all terms containing the specified variable to one side using the addition/subtraction property of equality.
- Step 3. Isolate the variable you are solving for using the multiplication/division property of equality.

EXAMPLE 1

Solve a Formula for a Specific Variable

The formula for the perimeter of a rectangle is found using the formula: $P = 2l + 2w$. Solve this formula in terms of l .



✓ Solution

$$P = 2l + 2w \quad \text{Since we are solving for } l \text{ we isolate the } l \text{ term}$$

$$P - 2w = 2l + 2w - 2w \quad \text{Subtract } 2w \text{ from both sides.}$$

$$P - 2w = 2l \quad \text{Combine like terms}$$

$$\frac{P-2w}{2} = \frac{2l}{2} \quad \text{Divide by 2 to isolate } l.$$

$$\frac{P-2w}{2} = l \quad \text{Simplify}$$

Practice Makes Perfect

Solve each formula for the specific variable.

1. Solve for b
 $P = a + b + c$

2. Solve for s

$$P = 4s$$

3. Solve for r

$$C = 2\pi r$$

4. Solve for b

$$A = \frac{1}{2}bh$$

5. Solve for W

$$P = 2L + 2W$$

6. Solve for m

$$y = mx + b$$

7. Solve for h

$$A = 2\pi h + 2\pi r^2$$

8. Solve for r

$$A = \pi r^2$$

9. Solve for s

$$V = \frac{1}{3}s^2h$$

10. Solve for L

$$A = 2LW + 2HW + 2LH$$

Objective 2: Use a problem-solving strategy for word problems (IA 2.2.1)



HOW TO

Use a Problem-Solving Strategy for word problems.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using proper algebra techniques.

Step 6. **Check** the answer in the problem to make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

EXAMPLE 2

Use a Problem-Solving Strategy for word problems.

Hang borrowed \$7,500 from her parents to pay her tuition. In five years, she paid them \$1,500 interest in addition to the \$7,500 she borrowed. What was the rate of simple interest?

✔ **Solution**

Write down the given information:	$I = \$1500$
	$P = \$7500$
	$r = ?$
	$t = 5 \text{ years}$
Identify the unknown: let interest rate be represented by r	
Write a formula:	$I = Prt$
Substitute in the given information:	$1500 = (7500)r(5)$
Solve for r	$1500 = 37,500r$
	$\frac{1500}{37,500} = r$
	$0.04 = r$
	$4\% = r$

Practice Makes Perfect

Use a Problem-Solving Strategy for word problems.

11. The formula for area of a trapezoid is $A = \frac{1}{2}(B + b)h$ where B is the length of the base, b is the length of the other base and h is the height of the trapezoid.



- If $B = 10\text{cm}$, $b = 8\text{cm}$ and $A = 45\text{cm}^2$, find the height of the trapezoid.
12. A married couple together earns \$110,000 a year. The wife earns \$16,000 less than twice what her husband earns. What does the husband earn?
13. The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?
14. Recently, the California governor proposed raising community college fees from \$36 a unit to \$46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)
15. Sean's new car loan statement said he would pay \$4,866.25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?
16. At the campus coffee cart, a medium coffee costs \$1.65. MaryAnne brings \$2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?



Figure 1 Credit: Kevin Dooley

Neka is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Neka to end the course with an A? A simple linear equation will give Neka his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as \$0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges \$0.10/mi plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost C .

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. [Table 1](#) lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by a	$x, x + a$
Twice a number	$2x$
One number is a more than another number	$x, x + a$
One number is a less than twice another number	$x, 2x - a$
The product of a number and a , decreased by b	$ax - b$
The quotient of a number and the number plus a is three times the number	$\frac{x}{x+a} = 3x$
The product of three times a number and the number decreased by b is c	$3x(x - b) = c$

Table 1



HOW TO

Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

EXAMPLE 1

Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x + 17$. The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.

$$\begin{aligned} x + (x + 17) &= 31 \\ 2x + 17 &= 31 && \text{Simplify and solve.} \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

$$\begin{aligned} x + 17 &= 7 + 17 \\ &= 24 \end{aligned}$$

The two numbers are 7 and 24.

TRY IT #1 Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36, find the numbers.

EXAMPLE 2

Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of \$34 plus \$.05/min talk-time. Company B charges a monthly service fee of \$40 plus \$.04/min talk-time.

- a) Write a linear equation that models the packages offered by both companies.
- b) If the average number of minutes used each month is 1,160, which company offers the better plan?
- c) If the average number of minutes used each month is 420, which company offers the better plan?
- d) How many minutes of talk-time would yield equal monthly statements from both companies?

Solution

a) The model for Company A can be written as $A = 0.05x + 34$. This includes the variable cost of $0.05x$ plus the monthly service charge of \$34. Company B's package charges a higher monthly fee of \$40, but a lower variable cost of $0.04x$. Company B's model can be written as $B = 0.04x + 40$.

ⓑ

If the average number of minutes used each month is 1,160, we have the following:

$$\begin{aligned}\text{Company } A &= 0.05(1,160) + 34 \\ &= 58 + 34 \\ &= 92\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(1,160) + 40 \\ &= 46.4 + 40 \\ &= 86.4\end{aligned}$$

So, Company *B* offers the lower monthly cost of \$86.40 as compared with the \$92 monthly cost offered by Company *A* when the average number of minutes used each month is 1,160.

ⓒ

If the average number of minutes used each month is 420, we have the following:

$$\begin{aligned}\text{Company } A &= 0.05(420) + 34 \\ &= 21 + 34 \\ &= 55\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(420) + 40 \\ &= 16.8 + 40 \\ &= 56.8\end{aligned}$$

If the average number of minutes used each month is 420, then Company *A* offers a lower monthly cost of \$55 compared to Company *B*'s monthly cost of \$56.80.

ⓓ

To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x, y) coordinates: At what point are both the x -value and the y -value equal? We can find this point by setting the equations equal to each other and solving for x .

$$\begin{aligned}0.05x + 34 &= 0.04x + 40 \\ 0.01x &= 6 \\ x &= 600\end{aligned}$$

Check the x -value in each equation.

$$\begin{aligned}0.05(600) + 34 &= 64 \\ 0.04(600) + 40 &= 64\end{aligned}$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See [Figure 2](#)

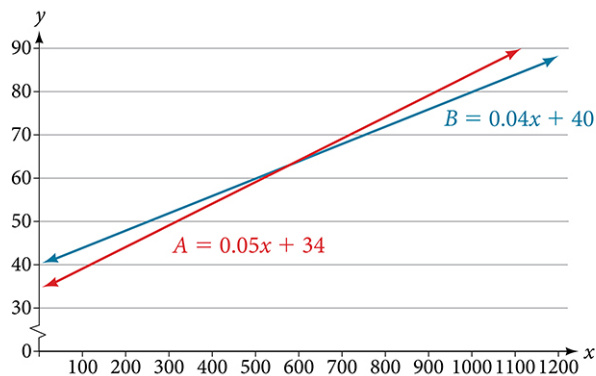


Figure 2

- > **TRY IT** #2 Find a linear equation to model this real-world application: It costs ABC electronics company \$2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of \$350 for utilities and \$3,300 for salaries. What are the company's monthly expenses?

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region, $A = LW$; the **perimeter** of a rectangle, $P = 2L + 2W$; and the **volume** of a rectangular solid, $V = LWH$. When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

EXAMPLE 3

Solving an Application Using a Formula

It takes Andrew 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

Solution

This is a distance problem, so we can use the formula $d = rt$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate r . His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance d . A table, such as [Table 2](#), is often helpful for keeping track of information in these types of problems.

	d	r	t
To Work	d	r	$\frac{1}{2}$
To Home	d	$r - 10$	$\frac{2}{3}$

Table 2

Write two equations, one for each trip.

$$\begin{aligned} d &= r\left(\frac{1}{2}\right) && \text{To work} \\ d &= (r - 10)\left(\frac{2}{3}\right) && \text{To home} \end{aligned}$$

As both equations equal the same distance, we set them equal to each other and solve for r .

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r - 10)\left(\frac{2}{3}\right) \\ \frac{1}{2}r &= \frac{2}{3}r - \frac{20}{3} \\ \frac{1}{2}r - \frac{2}{3}r &= -\frac{20}{3} \\ -\frac{1}{6}r &= -\frac{20}{3} \\ r &= -\frac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for d .

$$\begin{aligned} d &= 40\left(\frac{1}{2}\right) \\ &= 20 \end{aligned}$$

The distance between home and work is 20 mi.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r .

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6 \times r\left(\frac{1}{2}\right) &= 6 \times (r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ -r &= -40 \\ r &= 40 \end{aligned}$$

TRY IT #3 On Saturday morning, it took Jennifer 3.6 h to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer 4 h to return home. Her speed was 5 mi/h slower on Sunday than on Saturday. What was her speed on Sunday?

EXAMPLE 4

Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

Solution

The perimeter formula is standard: $P = 2L + 2W$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L = W + 3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in [Figure 3](#).

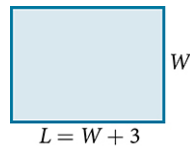


Figure 3

Now we can solve for the width and then calculate the length.

$$\begin{aligned} P &= 2L + 2W \\ 54 &= 2(W + 3) + 2W \\ 54 &= 2W + 6 + 2W \\ 54 &= 4W + 6 \\ 48 &= 4W \\ 12 &= W \\ (12 + 3) &= L \\ 15 &= L \end{aligned}$$

The dimensions are $L = 15$ ft and $W = 12$ ft.

TRY IT #4 Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

EXAMPLE 5**Solving an Area Problem**

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

✓ **Solution**

The standard formula for area is $A = LW$; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$\begin{aligned} P &= 2L + 2W \\ 48 &= 2(W + 6) + 2W \\ 48 &= 2W + 12 + 2W \\ 48 &= 4W + 12 \\ 36 &= 4W \\ 9 &= W \\ (9 + 6) &= L \\ 15 &= L \end{aligned}$$

Now, we find the area given the dimensions of $L = 15$ in. and $W = 9$ in.

$$\begin{aligned} A &= LW \\ A &= 15(9) \\ &= 135 \text{ in.}^2 \end{aligned}$$

The area is 135 in.².

> **TRY IT** #5 A game room has a perimeter of 70 ft. The length is five more than twice the width. How many ft² of new carpeting should be ordered?

EXAMPLE 6**Solving a Volume Problem**

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 in.³.

✓ **Solution**

The formula for the volume of a box is given as $V = LWH$, the product of length, width, and height. We are given that $L = 2W$, and $H = 8$. The volume is 1,600 cubic inches.

$$\begin{aligned} V &= LWH \\ 1,600 &= (2W)W(8) \\ 1,600 &= 16W^2 \\ 100 &= W^2 \\ 10 &= W \end{aligned}$$

The dimensions are $L = 20$ in., $W = 10$ in., and $H = 8$ in.

⊙ **Analysis**

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

 **MEDIA**

Access these online resources for additional instruction and practice with models and applications of linear equations.

[Problem solving using linear equations \(http://openstax.org/l/lineqprobsolve\)](http://openstax.org/l/lineqprobsolve)

[Problem solving using equations \(http://openstax.org/l/equationprsolve\)](http://openstax.org/l/equationprsolve)

[Finding the dimensions of area given the perimeter \(http://openstax.org/l/permareasolve\)](http://openstax.org/l/permareasolve)

[Find the distance between the cities using the distance = rate * time formula \(http://openstax.org/l/ratetimesolve\)](http://openstax.org/l/ratetimesolve)

[Linear equation application \(Write a cost equation\) \(http://openstax.org/l/lineqappl\)](http://openstax.org/l/lineqappl)



2.3 SECTION EXERCISES

Verbal

- To set up a model linear equation to fit real-world applications, what should always be the first step?
- Use your own words to describe this equation where n is a number:
 $5(n + 3) = 2n$
- If the total amount of money you had to invest was \$2,000 and you deposit x amount in one investment, how can you represent the remaining amount?
- If a carpenter sawed a 10-ft board into two sections and one section was n ft long, how long would the other section be in terms of n ?
- If Bill was traveling v mi/h, how would you represent Daemon's speed if he was traveling 10 mi/h faster?

Real-World Applications

For the following exercises, use the information to find a linear algebraic equation model to use to answer the question being asked.

- Mark and Don are planning to sell each of their marble collections at a garage sale. If Don has 1 more than 3 times the number of marbles Mark has, how many does each boy have to sell if the total number of marbles is 113?
- Beth and Ann are joking that their combined ages equal Sam's age. If Beth is twice Ann's age and Sam is 69 yr old, what are Beth and Ann's ages?
- Ruden originally filled out 8 more applications than Hanh. Then each boy filled out 3 additional applications, bringing the total to 28. How many applications did each boy originally fill out?

For the following exercises, use this scenario: Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of \$20 and charges of \$.05/min for calls. Company B has a monthly fee of \$5 and charges \$.10/min for calls.

- Find the model of the total cost of Company A's plan, using m for the minutes.
- Find the model of the total cost of Company B's plan, using m for the minutes.
- Find out how many minutes of calling would make the two plans equal.

12. If the person makes a monthly average of 200 min of calls, which plan should for the person choose?

For the following exercises, use this scenario: A wireless carrier offers the following plans that a person is considering. The Family Plan: \$90 monthly fee, unlimited talk and text on up to 8 lines, and data charges of \$40 for each device for up to 2 GB of data per device. The Mobile Share Plan: \$120 monthly fee for up to 10 devices, unlimited talk and text for all the lines, and data charges of \$35 for each device up to a shared total of 10 GB of data. Use P for the number of devices that need data plans as part of their cost.

13. Find the model of the total cost of the Family Plan.
14. Find the model of the total cost of the Mobile Share Plan.
15. Assuming they stay under their data limit, find the number of devices that would make the two plans equal in cost.
16. If a family has 3 smart phones, which plan should they choose?

For exercises 17 and 18, use this scenario: A retired woman has \$50,000 to invest but needs to make \$6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put in a CD that pays 7%.

17. If we let x be the amount the woman invests in the 15% bond, how much will she be able to invest in the CD?
18. Set up and solve the equation for how much the woman should invest in each option to sustain a \$6,000 annual return.
19. Two planes fly in opposite directions. One travels 450 mi/h and the other 550 mi/h. How long will it take before they are 4,000 mi apart?
20. Ben starts walking along a path at 4 mi/h. One and a half hours after Ben leaves, his sister Amanda begins jogging along the same path at 6 mi/h. How long will it be before Amanda catches up to Ben?
21. Fiora starts riding her bike at 20 mi/h. After a while, she slows down to 12 mi/h, and maintains that speed for the rest of the trip. The whole trip of 70 mi takes her 4.5 h. For what distance did she travel at 20 mi/h?
22. A chemistry teacher needs to mix a 30% salt solution with a 70% salt solution to make 20 qt of a 40% salt solution. How many quarts of each solution should the teacher mix to get the desired result?
23. Raúl has \$20,000 to invest. His intent is to earn 11% interest on his investment. He can invest part of his money at 8% interest and part at 12% interest. How much does Raúl need to invest in each option to make get a total 11% return on his \$20,000?

For the following exercises, use this scenario: A truck rental agency offers two kinds of plans. Plan A charges \$75/wk plus \$.10/mi driven. Plan B charges \$100/wk plus \$.05/mi driven.

24. Write the model equation for the cost of renting a truck with plan A.
25. Write the model equation for the cost of renting a truck with plan B.
26. Find the number of miles that would generate the same cost for both plans.
27. If Tim knows he has to travel 300 mi, which plan should he choose?

For the following exercises, use the formula given to solve for the required value.

28. $A = P(1 + rt)$ is used to find the principal amount P deposited, earning $r\%$ interest, for t years. Use this to find what principal amount P David invested at a 3% rate for 20 yr if $A = \$8,000$.
29. The formula $F = \frac{mv^2}{R}$ relates force (F), velocity (v), mass, and resistance (m). Find R when $m = 45$, $v = 7$, and $F = 245$.
30. $F = ma$ indicates that force (F) equals mass (m) times acceleration (a). Find the acceleration of a mass of 50 kg if a force of 12 N is exerted on it.
31. $Sum = \frac{1}{1-r}$ is the formula for an infinite series sum. If the sum is 5, find r .

For the following exercises, solve for the given variable in the formula. After obtaining a new version of the formula, you will use it to solve a question.

32. Solve for W : $P = 2L + 2W$
33. Use the formula from the previous question to find the width, W , of a rectangle whose length is 15 and whose perimeter is 58.
34. Solve for f : $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$
35. Use the formula from the previous question to find f when $p = 8$ and $q = 13$.
36. Solve for m in the slope-intercept formula: $y = mx + b$
37. Use the formula from the previous question to find m when the coordinates of the point are (4, 7) and $b = 12$.
38. The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$. Use the formula to find the area of a trapezoid with $h = 6$, $b_1 = 14$, and $b_2 = 8$.
39. Solve for h : $A = \frac{1}{2}h(b_1 + b_2)$
40. Use the formula from the previous question to find the height of a trapezoid with $A = 150$, $b_1 = 19$, and $b_2 = 11$.

41. Find the dimensions of an American football field. The length is 200 ft more than the width, and the perimeter is 1,040 ft. Find the length and width. Use the perimeter formula $P = 2L + 2W$.
42. Distance equals rate times time, $d = rt$. Find the distance Tom travels if he is moving at a rate of 55 mi/h for 3.5 h.
43. Using the formula in the previous exercise, find the distance that Susan travels if she is moving at a rate of 60 mi/h for 6.75 h.
44. What is the total distance that two people travel in 3 h if one of them is riding a bike at 15 mi/h and the other is walking at 3 mi/h?
45. If the area model for a triangle is $A = \frac{1}{2}bh$, find the area of a triangle with a height of 16 in. and a base of 11 in.
46. Solve for h : $A = \frac{1}{2}bh$
47. Use the formula from the previous question to find the height to the nearest tenth of a triangle with a base of 15 and an area of 215.
48. The volume formula for a cylinder is $V = \pi r^2 h$. Using the symbol π in your answer, find the volume of a cylinder with a radius, r , of 4 cm and a height of 14 cm.
49. Solve for h : $V = \pi r^2 h$
50. Use the formula from the previous question to find the height of a cylinder with a radius of 8 and a volume of 16π
51. Solve for r : $V = \pi r^2 h$
52. Use the formula from the previous question to find the radius of a cylinder with a height of 36 and a volume of 324π .
53. The formula for the circumference of a circle is $C = 2\pi r$. Find the circumference of a circle with a diameter of 12 in. (diameter = $2r$). Use the symbol π in your final answer.
54. Solve the formula from the previous question for π . Notice why π is sometimes defined as the ratio of the circumference to its diameter.

2.4 Complex Numbers

Learning Objectives

In this section, you will:

- Add and subtract complex numbers.
- Multiply and divide complex numbers.
- Simplify powers of i

COREQUISITE SKILLS

Learning Objectives

- Use the product property to simplify radical expressions (IA 8.2.1)
- Evaluate the square root of a negative number (IA 8.8.1)

1.2 Functions: Modeling Relationships

Motivating Questions

- How can we use the mathematical idea of a function to represent the relationship between two changing quantities?
 - What are some formal characteristics of an abstract mathematical function? how do we think differently about these characteristics in the context of a physical model?
-

A mathematical model is an abstract concept through which we use mathematical language and notation to describe a phenomenon in the world around us. One example of a mathematical model is found in *Dolbear's Law*¹. In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. If we let T represent the temperature in degrees Fahrenheit and N the number of chirps per minute, we can summarize Dolbear's observations in the following table.

N (chirps per minute)	40	80	120	160
T (° Fahrenheit)	50°	60°	70°	80°

Table 1.2.1: Data for Dolbear's observations.

For a mathematical model, we often seek an algebraic formula that captures observed behavior accurately and can be used to predict behavior not yet observed. For the data in Table 1.2.1, we observe that each of the ordered pairs in the table make the equation

$$T = 40 + 0.25N \tag{1.2.1}$$

true. For instance, $70 = 40 + 0.25(120)$. Indeed, scientists who made many additional cricket chirp observations following Dolbear's initial counts found that the formula in Equation (1.2.1) holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from about 50° F to 85° F.

Preview Activity 1.2.1. Use Equation (1.2.1) to respond to the questions below.

- If we hear snowy tree crickets chirping at a rate of 92 chirps per minute, what does Dolbear's model suggest should be the outside temperature?
- If the outside temperature is 77° F, how many chirps per minute should we expect to hear?
- Is the model valid for determining the number of chirps one should hear when the outside temperature is 35° F? Why or why not?

¹You can read more in the Wikipedia entry for Dolbear's Law, which has proven to be remarkably accurate for the behavior of snowy tree crickets. For even more of the story, including a reference to this phenomenon on the popular show *The Big Bang Theory*, see this article.

- d. Suppose that in the morning an observer hears 65 chirps per minute, and several hours later hears 75 chirps per minute. How much has the temperature risen between observations?
- e. Dolbear's Law is known to be accurate for temperatures from 50° to 85° . What is the fewest number of chirps per minute an observer could expect to hear? the greatest number of chirps per minute?

1.2.1 Functions

The mathematical concept of a *function* is one of the most central ideas in all of mathematics, in part since functions provide an important tool for representing and explaining patterns. At its core, a function is a repeatable process that takes a collection of input values and generates a corresponding collection of output values with the property that if we use a particular single input, the process always produces exactly the same single output.

For instance, Dolbear's Law in Equation (1.2.1) provides a process that takes a given number of chirps between 40 and 180 per minute and reliably produces the corresponding temperature that corresponds to the number of chirps, and thus this equation generates a function. We often give functions shorthand names; using "*D*" for the "Dolbear" function, we can represent the process of taking inputs (observed chirp rates) to outputs (corresponding temperatures) using arrows:

$$\begin{aligned} 80 &\xrightarrow{D} 60 \\ 120 &\xrightarrow{D} 70 \\ N &\xrightarrow{D} 40 + 0.25N \end{aligned}$$

Alternatively, for the relationship " $80 \xrightarrow{D} 60$ " we can also use the equivalent notation " $D(80) = 60$ " to indicate that Dolbear's Law takes an input of 80 chirps per minute and produces a corresponding output of 60 degrees Fahrenheit. More generally, we write " $T = D(N) = 40 + 0.25N$ " to indicate that a certain temperature, T , is determined by a given number of chirps per minute, N , according to the process $D(N) = 40 + 0.25N$.

Tables and graphs are particularly valuable ways to characterize and represent functions. For the current example, we summarize some of the data the Dolbear function generates in Table 1.2.2 and plot that data along with the underlying curve in Figure 1.2.3.

When a point such as $(120, 70)$ in Figure 1.2.3 lies on a function's graph, this indicates the correspondence between input and output: when the value 120 chirps per minute is entered in the function D , the result is 70 degrees Fahrenheit. More concisely, $D(120) = 70$. Aloud, we read " D of 120 is 70".

For most important concepts in mathematics, the mathematical community decides on formal definitions to ensure that we have a shared language of understanding. In this text, we will use the following definition of the term "function".

N	T
40	50
80	60
120	70
160	80
180	85

Table 1.2.2: Data for the function $T = D(N) = 40 + 0.25N$.

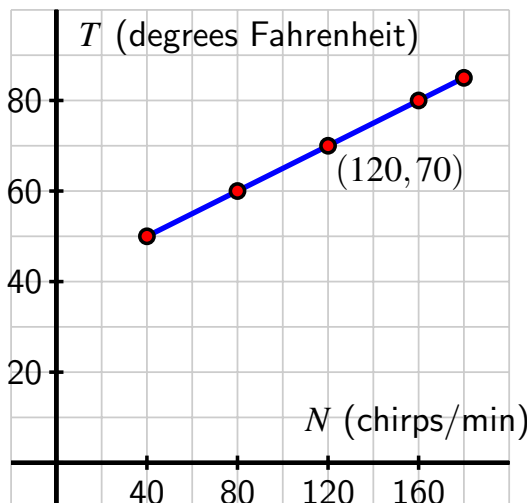


Figure 1.2.3: Graph of data from the function $T = D(N) = 40 + 0.25N$ and the underlying curve.

Definition 1.2.4 A **function** is a process that may be applied to a collection of input values to produce a corresponding collection of output values in such a way that the process produces one and only one output value for any single input value. \diamond

If we name a given function F and call the collection of possible inputs to F the set A and the corresponding collection of potential outputs B , we say “ F is a function from A to B ,” and sometimes write “ $F : A \rightarrow B$.” When a particular input value to F , say t , produces a corresponding output z , we write “ $F(t) = z$ ” and read this symbolic notation as “ F of t is z .” We often call t the *independent variable* and z the *dependent variable*, since z is a function of t .

Definition 1.2.5 Let F be a function from A to B . The set A of possible inputs to F is called the **domain** of F ; the set B of potential outputs from F is called the **codomain** of F . \diamond

For the Dolbear function $D(N) = 40 + 0.25N$ in the context of modeling temperature as a function of the number of cricket chirps per minute, the domain of the function is $A = [40, 180]^2$ and the codomain is “all Fahrenheit temperatures”. The codomain of a function is the collection of *possible* outputs, which we distinguish from the collection of *actual* outputs.

Definition 1.2.6 Let F be a function from A to B . The **range** of F is the collection of all actual outputs of the function. That is, the range is the collection of all elements y in B for which it is possible to find an element x in A such that $F(x) = y$. \diamond

In many situations, the range of a function is much more challenging to determine than its codomain. For the Dolbear function, the range is straightforward to find by using the graph shown in Figure 1.2.3: since the actual outputs of D fall between $T = 50$ and $T = 85$ and

²The notation “[40, 180]” means “the collection of all real numbers x that satisfy $40 \leq x \leq 180$ ” and is sometimes called “interval notation”.

include every value in that interval, the range of D is $[50, 80]$.

The range of any function is always a subset of the codomain. It is possible for the range to equal the codomain.

Activity 1.2.2. Consider a spherical tank of radius 4 m that is filling with water. Let V be the volume of water in the tank (in cubic meters) at a given time, and h the depth of the water (in meters) at the same time. It can be shown using calculus that V is a function of h according to the rule

$$V = f(h) = \frac{\pi}{3}h^2(12 - h).$$

- What values of h make sense to consider in the context of this function? What values of V make sense in the same context?
- What is the domain of the function f in the context of the spherical tank? Why? What is the corresponding codomain? Why?
- Determine and interpret (with appropriate units) the values $f(2)$, $f(4)$, and $f(8)$. What is important about the value of $f(8)$?
- Consider the claim: “since $f(9) = \frac{\pi}{3}9^2(12 - 9) = 81\pi \approx 254.47$, when the water is 9 meters deep, there is about 254.47 cubic meters of water in the tank”. Is this claim valid? Why or why not? Further, does it make sense to observe that “ $f(13) = -\frac{169\pi}{3}$ ”? Why or why not?
- Can you determine a value of h for which $f(h) = 300$ cubic meters?

1.2.2 Comparing models and abstract functions

Again, a mathematical model is an abstract concept through which we use mathematical language and notation to describe a phenomenon in the world around us. So far, we have considered two different examples: the Dolbear function, $T = D(N) = 40 + 0.25N$, that models how Fahrenheit temperature is a function of the number of cricket chirps per minute and the function $V = f(h) = \frac{\pi}{3}h^2(12 - h)$ that models how the volume of water in a spherical tank of radius 4 m is a function of the depth of the water in the tank. While often we consider a function in the physical setting of some model, there are also many occasions where we consider an abstract function for its own sake in order to study and understand it.

Example 1.2.7 A parabola and a falling ball. Calculus shows that for a tennis ball tossed vertically from a window 48 feet above the ground at an initial vertical velocity of 32 feet per second, the ball’s height above the ground at time t (where $t = 0$ is the instant the ball is tossed) can be modeled by the function $h = g(t) = -16t^2 + 32t + 48$. Discuss the differences between the model g and the abstract function f determined by $y = f(x) = -16x^2 + 32x + 48$.

Solution. We start with the abstract function $y = f(x) = -16x^2 + 32x + 48$. Absent a physical context, we can investigate the behavior of this function by computing function values, plotting points, and thinking about its overall behavior. We recognize the function

f as quadratic³, noting that it opens down because of the leading coefficient of -16 , with vertex located at $x = \frac{-32}{2(-16)} = 1$, y -intercept at $(0, 48)$, and with x -intercepts at $(-1, 0)$ and $(3, 0)$ because

$$-16x^2 + 32x + 48 = -16(x^2 - 2x - 3) = -16(x - 3)(x + 1).$$

Computing some additional points to gain more information, we see both the data in Table 1.2.8 and the corresponding graph in Figure 1.2.9.

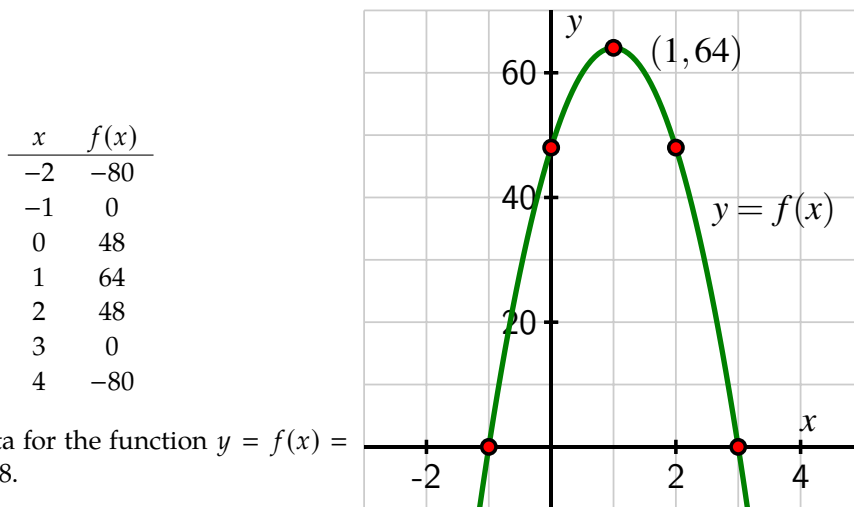


Table 1.2.8: Data for the function $y = f(x) = -16x^2 + 32x + 48$.

Figure 1.2.9: Graph of the function $y = f(x)$ and some data from the table.

For this abstract function, its domain is “all real numbers” since we may input any real number x we wish into the formula $f(x) = -16x^2 + 32x + 48$ and have the result be defined. Moreover, taking a real number x and processing it in the formula $f(x) = -16x^2 + 32x + 48$ will produce another real number. This tells us that the codomain of the abstract function f is also “all real numbers.” Finally, from the graph and the data, we observe that the largest possible output of the function f is $y = 64$. It is apparent that we can generate any y -value less than or equal to 64, and thus the range of the abstract function f is all real numbers less than or equal to 64. We denote this collection of real numbers using the shorthand interval notation $(-\infty, 64]$.⁴

Next, we turn our attention to the model $h = g(t) = -16t^2 + 32t + 48$ that represents the height of the ball, h , in feet t seconds after the ball is initially launched. Here, the big difference is the domain, codomain, and range associated with the model. Since the model takes effect once the ball is tossed, it only makes sense to consider the model for input values $t \geq 0$. Moreover, because the model ceases to apply once the ball lands, it is only valid for $t \leq 3$. Thus, the domain of g is $[0, 3]$. For the codomain, it only makes sense to consider values of h that are nonnegative. That is, as we think of *potential* outputs for the model, then can only be in the interval $[0, \infty)$. Finally, we can consider the graph of the model on the given domain in Figure 1.2.11 and see that the range of the model is $[0, 64]$, the collection of all heights between its lowest (ground level) and its largest (at the vertex).

t	$g(t)$
0	48
1	64
2	48
3	0

Table 1.2.10: Data for the model $h = g(t) = -16t^2 + 32t + 48$.

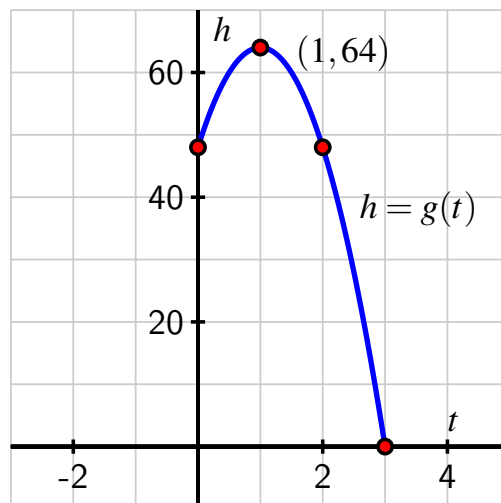


Figure 1.2.11: Graph of the model $h = g(t)$ and some data from the table.

□

Activity 1.2.3. Consider a spherical tank of radius 4 m that is completely full of water. Suppose that the tank is being drained by regulating an exit valve in such a way that the height of the water in the tank is always decreasing at a rate of 0.5 meters per minute. Let V be the volume of water in the tank (in cubic meters) at a given time t (in minutes), and h the depth of the water (in meters) at the same time. It can be shown using calculus that V is a function of t according to the model

$$V = p(t) = \frac{256\pi}{3} - \frac{\pi}{24}t^2(24 - t).$$

In addition, let $h = q(t)$ be the function whose output is the depth of the water in the tank at time t .

- What is the height of the water when $t = 0$? When $t = 1$? When $t = 2$? How long will it take the tank to completely drain? Why?
- What is the domain of the model $h = q(t)$? What is the domain of the model $V = p(t)$?
- How much water is in the tank when the tank is full? What is the range of the model $h = q(t)$? What is the range of the model $V = p(t)$?
- We will frequently use a graphing utility to help us understand function behavior, and strongly recommend *Desmos* because it is intuitive, online, and free.⁵

In this prepared *Desmos* worksheet, you can see how we enter the (abstract) function $V = p(t) = \frac{256\pi}{3} - \frac{\pi}{24}t^2(24 - t)$, as well as the corresponding graph

³We will engage in a brief review of quadratic functions in Section 1.5

⁴The notation $(-\infty, 64]$ stands for all the real numbers that lie to the left of and including 64. The “ $-\infty$ ” indicates that there is no left-hand bound on the interval.

the program generates. Make as many observations as you can about the model $V = p(t)$. You should discuss its shape and overall behavior, its domain, its range, and more.

- e. How does the model $V = p(t) = \frac{256\pi}{3} - \frac{\pi}{24}t^2(24 - t)$ differ from the abstract function $y = r(x) = \frac{256\pi}{3} - \frac{\pi}{24}x^2(24 - x)$? In particular, how do the domain and range of the model differ from those of the abstract function, if at all?
- f. How should the graph of the height function $h = q(t)$ appear? Can you determine a formula for q ? Explain your thinking.

1.2.3 Determining whether a relationship is a function or not

To this point in our discussion of functions, we have mostly focused on what the function process may model and what the domain, codomain, and range of a model or abstract function are. It is also important to take note of another part of Definition 1.2.4: "... the process produces one and only one output value for any single input value". Said differently, if a relationship or process ever associates a single input with two or more different outputs, the process cannot be a function.

Example 1.2.12 Is the relationship between people and phone numbers a function?

Solution. No, this relationship is not a function. A given individual person can be associated with more than one phone number, such as their cell phone and their work telephone. This means that we can't view phone numbers as a function of people: one input (a person) can lead to two different outputs (phone numbers). We also can't view people as a function of phone numbers, since more than one person can be associated with a phone number, such as when a family shares a single phone at home. \square

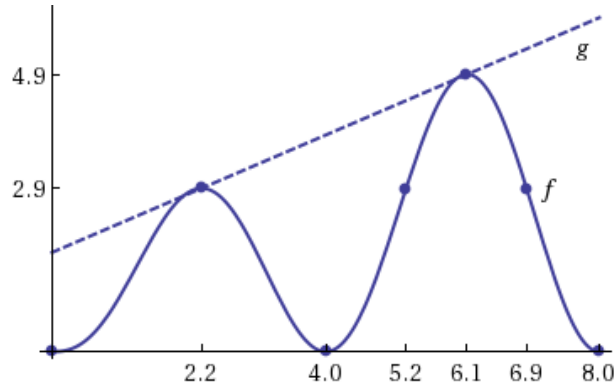
Example 1.2.13 The relationship between x and y that is given in the following table where we attempt to view y as depending on x .

x	1	2	3	4	5
y	13	11	10	11	13

Table 1.2.14: A table that relates x and y values.

Solution. The relationship between y and x in Table 1.2.14 allows us to think of y as a function of x since each particular input is associated with one and only one output. If we name the function f , we can say for instance that $f(4) = 11$. Moreover, the domain of f is the set of inputs $\{1, 2, 3, 4, 5\}$, and the codomain (which is also the range) is the set of outputs $\{10, 11, 13\}$. \square

⁵To learn more about *Desmos*, see their outstanding online tutorials.



- (a) Find $f(5.2)$.
- (b) Fill in the blanks in each of the two points below to correctly complete the coordinates of two points on the graph of $g(x)$.
- (6.1, _____) (_____, 2.9)
- (c) For what value(s) of x is/are $f(x) = 2.9$?
- (d) For what value(s) of x is/are $f(x) = g(x)$?
2. The table below $A = f(d)$, the amount of money A (in billions of dollars) in bills of denomination d circulating in US currency in 2005. For example according to the table values below there were \$60.2 billion worth of \$50 bills in circulation.



Denomination (value of bill)	1	5	10	20	50	100
Dollar Value in Circulation	8.4	9.7	14.8	110.1	60.2	524.5

- a) Find $f(20)$.
- b) Using your answer in (a), what was the total **number of \$20 bills** (not amount of money) in circulation in 2005?
- c) Are the following statements True or False?
- (i) There were more 20 dollar bills than 100 dollar bills
- (ii) There were more 5 dollar bills than 20 dollar bills
3. Let $f(t)$ denote the number of people eating in a restaurant t minutes after 5 PM. Answer the following questions:
- a) Which of the following statements best describes the significance of the expression $f(4) = 21$?



- There are 4 people eating at 5:21 PM
- There are 21 people eating at 5:04 PM
- There are 21 people eating at 9:00 PM
- Every 4 minutes, 21 more people are eating
- None of the above

b) Which of the following statements best describes the significance of the expression $f(a) = 20$?

- a minutes after 5 PM there are 20 people eating
- Every 20 minutes, the number of people eating has increased by a people
- At 5:20 PM there are a people eating
- a hours after 5 PM there are 20 people eating
- None of the above

c) Which of the following statements best describes the significance of the expression $f(20) = b$?

- Every 20 minutes, the number of people eating has increased by b people
- b minutes after 5 PM there are 20 people eating
- At 5:20 PM there are b people eating
- b hours after 5 PM there are 20 people eating
- None of the above

d) Which of the following statements best describes the significance of the expression $n = f(t)$?

- Every t minutes, n more people have begun eating
- n hours after 5 PM there are t people eating
- n minutes after 5 PM there are t people eating
- t hours after 5 PM there are n people eating
- None of the above



4. Chicago's average monthly rainfall, $R = f(t)$ inches, is given as a function of the month, t , where January is $t = 1$, in the table below.

t , month	1	2	3	4	5	6	7	8
R , inches	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4

(a) Solve $f(t) = 3.4$.

The solution(s) to $f(t) = 3.4$ can be interpreted as saying

- Chicago's average rainfall is least in the month of August.
- Chicago's average rainfall in the month of August is 3.4 inches.
- Chicago's average rainfall increases by 3.4 inches in the month of May.

- ⊙ Chicago's average rainfall is greatest in the month of May.
- ⊙ None of the above

(b) Solve $f(t) = f(5)$.

The solution(s) to $f(t) = f(5)$ can be interpreted as saying

- ⊙ Chicago's average rainfall is greatest in the month of May.
- ⊙ Chicago's average rainfall is 3.5 inches in the months of May and July.
- ⊙ Chicago's average rainfall is 3.5 inches in the month of May.
- ⊙ Chicago's average rainfall is 3.5 inches in the month of July.
- ⊙ None of the above

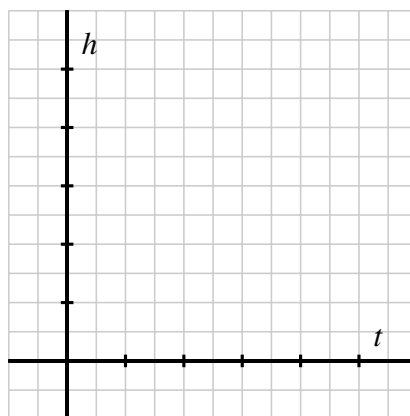
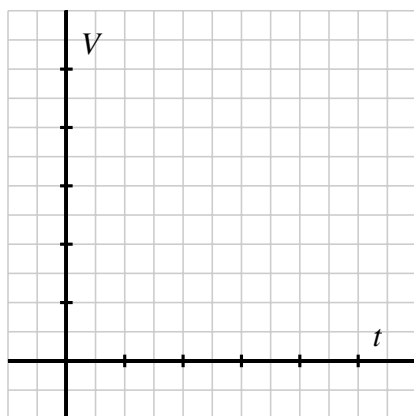
5. A national park records data regarding the total fox population F over a 12 month period, where $t = 0$ means January 1, $t = 1$ means February 1, and so on. Below is the table of values they recorded:



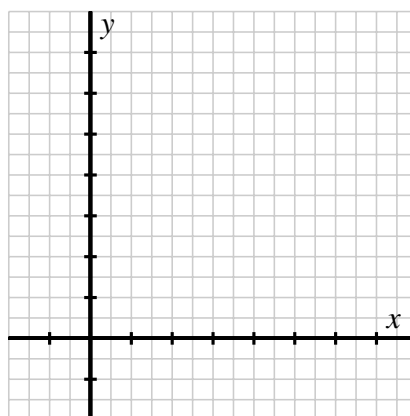
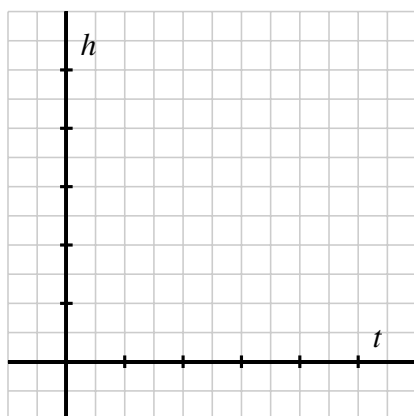
t, month	0	1	2	3	4	5	6	7	8	9	10	11
F, foxes	150	143	125	100	75	57	50	57	75	100	125	143

- (a) Is t a function of F ?
- (b) Let $g(t) = F$ denote the fox population in month t . Find all solution(s) to the equation $g(t) = 125$. If there is more than one solution, give your answer as a comma separated list of numbers.
6. An open box is to be made from a flat piece of material 20 inches long and 6 inches wide by cutting equal squares of length x from the corners and folding up the sides.
- Write the volume V of the box as a function of x . Leave it as a product of factors, do not multiply out the factors.
- If we write the domain of the box as an open interval in the form (a, b) , then what is a and what is b ?
7. Consider an inverted conical tank (point down) whose top has a radius of 3 feet and that is 2 feet deep. The tank is initially empty and then is filled at a constant rate of 0.75 cubic feet per minute. Let $V = f(t)$ denote the volume of water (in cubic feet) at time t in minutes, and let $h = g(t)$ denote the depth of the water (in feet) at time t .
- a. Recall that the volume of a conical tank of radius r and depth h is given by the formula $V = \frac{1}{3}\pi r^2 h$. How long will it take for the tank to be completely full and how much water will be in the tank at that time?
 - b. On the provided axes, sketch possible graphs of both $V = f(t)$ and $h = g(t)$, making them as accurate as you can. Label the scale on your axes and points whose coordinates you know for sure; write at least one sentence for each graph to discuss the shape of your graph and why it makes sense in the context of the model.





- c. What is the domain of the model $h = g(t)$? its range? why?
- d. It's possible to show that the formula for the function g is $g(t) = \left(\frac{t}{\pi}\right)^{1/3}$. Use a computational device to generate two plots: on the axes at left, the graph of the model $h = g(t) = \left(\frac{t}{\pi}\right)^{1/3}$ on the domain that you decided in (c); on the axes at right, the graph of the abstract function $y = p(x) = \left(\frac{t}{\pi}\right)^{1/3}$ on a wider domain than that of g . What are the domain and range of p and how do these differ from those of the physical model g ?



8. A person is taking a walk along a straight path. Their velocity, v (in feet per second), which is a function of time t (in seconds), is given by the graph in Figure 1.2.19.
- What is the person's velocity when $t = 2$? when $t = 7$?
 - Are there any times when the person's velocity is exactly $v = 3$ feet per second? If yes, identify all such times; if not, explain why.
 - Describe the person's behavior on the time interval $4 \leq t \leq 5$.

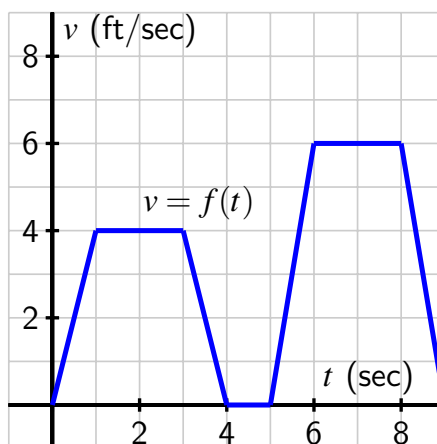


Figure 1.2.19: The velocity graph for a person walking along a straight path.

- d. On which time interval does the person travel a farther distance: $[1, 3]$ or $[6, 8]$? Why?
9. A driver of a new car periodically keeps track of the number of gallons of gas remaining in their car's tank, while simultaneously tracking the trip odometer mileage. Their data is recorded in the following table. Note that at mileages where they add fuel to the tank, they record the mileage twice: once before fuel is added, and once afterward.

D (miles)	0	50	100	100	150	200	250	300	300	350
G (gallons)	4.5	3.0	1.5	10.0	8.5	7.0	5.5	4.0	11.0	9.5

Table 1.2.20: Remaining gas as a function of distance traveled.

Use the table to respond to the questions below.

- Can the amount of fuel in the gas tank, G , be viewed as a function of distance traveled, D ? Why or why not?
- Does the car's fuel economy appear to be constant or does it appear to vary? Why?
- At what odometer reading did the driver put the most gas in the tank?

Activity 3.7.8 Sketch the graph of each of the following functions using the guide to curve sketching found in [Remark 3.7.7](#)

(a) $f(x) = x^4 - 4x^3 + 10$

(b) $f(x) = \frac{x^2-4}{x^2-9}$

(c) $f(x) = x + 2 \cos x$ on the interval $[0, 2\pi]$

(d) $f(x) = \frac{x^2+x-2}{x+3}$

(e) $f(x) = \frac{x}{\sqrt{x^2+2}}$

(f) $f(x) = x^6 + \frac{12}{5}x^5 - 12x^4 + 10$

3.7.2 Videos

No video is available for this learning outcome.

3.8 Applied optimization (AD8)

Learning Outcomes

- Apply optimization techniques to solve various problems.

3.8.1 Activities

Activity 3.8.1 The box. Help your company design an open box (no lid) with maximum volume given the following constraints:

- The box must be made from the following material: an 8 by 8 inches piece of cardboard.
- To create the box, you are asked to cut out a square from each corner of the 8 by 8 inches piece of cardboard and to fold up the flaps to create the sides.

- (a) Draw a diagram illustrating how the box is created.
- (b) Explain why the volume of the box is a function of the side length x of the cutout squares.
- (c) Express the volume of the box V as a function of the length of the cuts x .
- (d) What is a realistic domain of the function $V(x)$?
- (e) What cut length x maximizes the volume of the box?

Remark 3.8.2 A guide for optimization problems.

1. Draw a diagram and introduce variables.
2. Determine a function of a single variable that models the quantity to be optimized.
3. Decide the domain on which to consider the function being optimized.
4. Use calculus to identify the global maximum and/or minimum of the quantity being optimized.

5. Conclusion: what are the optimal points and what optimal values do we obtain at these points?

Activity 3.8.3 According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where the “girth” is the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

- (a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in [Figure 76](#).

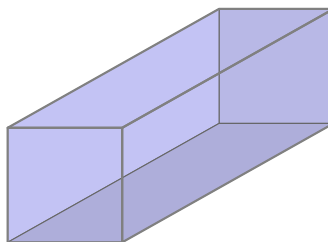


Figure 76 A rectangular parcel with a square end.

- (b) What is the quantity to be optimized in this problem?
- maximize volume (call this V)
 - maximize the girth plus length (call this P)
 - minimize volume (call this V)
 - minimize the girth plus length (call this P)
- (c) Which formula below represents the quantity you want to optimize in terms of x and y ?
- $V = x^2y$
 - $V = xy^2$
 - $P = 2x + y$
 - $P = 4x + y$
- (d) The problem statement tells us that the parcel’s girth plus length (P) may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length P to equal 108 inches. What equation does this constraint give us involving x and y ?
- $108 = 4x + y$
 - $108 = 2x + y$
 - $108 = x^2 + y$
 - $108 = xy^2$
- (e) The equation above gives the relationship between x and y . For ease of notation, solve this equation for y as a function on x and then find a formula for the volume of the parcel as a function of the single variable x . What is the formula for $V(x)$?
- $V(x) = x^2(108 - 4x)$

B. $V(x) = x(108 - 4x)^2$

C. $V(x) = x^2(108 - 2x)$

D. $V(x) = x(108 - 2x)^2$

- (f) Over what domain should we consider this function? To answer this question, notice that the problem gives us the constraint that P (girth plus length) is 108 inches. This constraint produces intervals of possible values for x and y .

A. $0 \leq x \leq 108$

B. $0 \leq y \leq 108$

C. $0 \leq x \leq 27$

D. $0 \leq y \leq 27$

- (g) Use *calculus* to find the global maximum of the volume of the parcel on the domain you just determined. Justify that you have found the global maximum using either the Closed Interval Method, the First Derivative Test, or the Second Derivative Test!

Remark 3.8.4 Notice that a critical point might or might not be an global maximum or minimum, so just finding the critical points is not enough to answer an optimization problem. Moreover, some of the critical points might be outside of the domain imposed by the context and thus they cannot be feasible optimal points.

Activity 3.8.5 Revenue = Number of tickets \times Price of ticket. Waterford movie theater currently charges \$8 for a ticket. At this price, the theater sells 200 tickets daily. The general manager wonders if they can generate more revenue by increasing the price of a tickets. A survey shows that they will lose 20 customers for every dollar increase in the ticket price.

- (a) If the price of a movie ticket is increased by d dollars, write a formula for the price P in terms of d .
- (b) If the price of a ticket is increased by one dollar, how many many customers will the theater lose?
- (c) Write a formula for the number of tickets sold T as a function of a price increase of d dollars.
- (d) Consider the new price of a ticket $P(d)$ and the new number of tickets sold $T(d)$. Write a formula for the revenue earned by ticket sales $R(d)$ as a function of a price increase of d dollars.
- (e) What is a realistic domain for the function $R(d)$?
- (f) What increase in price d should the general manager choose to maximize the revenue? What price would a movie ticket cost then and what would the revenue be at that price?
- (g) Suppose now that the cost of running the business when the price is increased by d dollars is given by $C(d) = 10d^3 - 40d^2 + 40d + 600$. If the manager decides that they will definitely increase the price, what price increase d maximizes the profit? (Recall that Profit = Revenue - Cost).

Activity 3.8.6 Modeling given a geometric shape. The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter

ter.

- What choice of dimensions will make the rectangular area in the center as large as possible?
- What should the dimensions so the total area enclosed by the running track is maximized?

Activity 3.8.7 Modeling in algebraic situations.

- Find the coordinates of the point on the curve $y = \sqrt{x}$ closest to the point $(1, 0)$.
- The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

Activity 3.8.8 Suppose that if a widget is priced at \$176, then you are able to sell 672 units each day. According to a survey of customers, increasing this price by \$1 will result in losing 4 daily sales; decreasing by \$1 will gain 4 daily sales. Your manager asks you how to adjust the price of a widget to maximize the revenue (widgets sold times price). Write an explanation of what this change in price should be and why.

3.8.2 Videos



YouTube: <https://www.youtube.com/watch?v=rpoDVt0nVeQ>

Figure 77 Video for AD8



YouTube: <https://www.youtube.com/watch?v=nLxzVHnzTKw>

Figure 78 Another Video for AD8

3.9 Limits and Derivatives (AD9)

Learning Outcomes

- Compute the values of indeterminate limits using L'Hopital's Rule.

5.2 circles, traversing a circle, circular functions (Active Prelude to Calculus)

Circular Functions

2.1 Traversing Circles

Motivating Questions

- How does a point traversing a circle naturally generate a function?
- What are some important properties that characterize a function generated by a point traversing a circle?
- How does a circular function change in ways that are different from linear and quadratic functions?

Certain naturally occurring phenomena eventually repeat themselves, especially when the phenomenon is somehow connected to a circle. For example, suppose that you are taking a ride on a ferris wheel and we consider your height, h , above the ground and how your height changes in tandem with the distance, d , that you have traveled around the wheel. In Figure 2.1.1 we see a snapshot of this situation, which is available as a full animation¹ at <http://gvsu.edu/s/0Dt>.

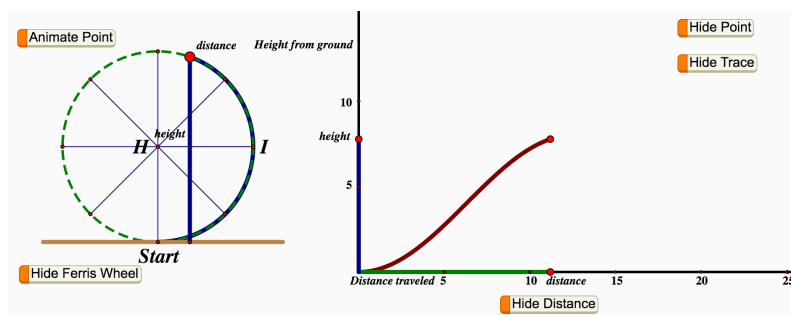


Figure 2.1.1: A snapshot of the motion of a cab moving around a ferris wheel. Reprinted with permission from Illuminations by the National Council of Teachers of Mathematics. All rights reserved.

¹Used with permission from Illuminations by the National Council of Teachers of Mathematics. All rights reserved.

Because we have two quantities changing in tandem, it is natural to wonder if it is possible to represent one as a function of the other.

Preview Activity 2.1.1. In the context of the ferris wheel pictured in Figure 2.1.1, assume that the height, h , of the moving point (the cab in which you are riding), and the distance, d , that the point has traveled around the circumference of the ferris wheel are both measured in meters.

Further, assume that the circumference of the ferris wheel is 150 meters. In addition, suppose that after getting in your cab at the lowest point on the wheel, you traverse the full circle several times.

- Recall that the circumference, C , of a circle is connected to the circle's radius, r , by the formula $C = 2\pi r$. What is the radius of the ferris wheel? How high is the highest point on the ferris wheel?
- How high is the cab after it has traveled $1/4$ of the circumference of the circle?
- How much distance along the circle has the cab traversed at the moment it first reaches a height of $\frac{150}{\pi} \approx 47.75$ meters?
- Can h be thought of as a function of d ? Why or why not?
- Can d be thought of as a function of h ? Why or why not?
- Why do you think the curve shown at right in Figure 2.1.1 has the shape that it does? Write several sentences to explain.

2.1.1 Circular Functions

The natural phenomenon of a point moving around a circle leads to interesting relationships. For easier arithmetic, let's consider a point traversing a circle of circumference 24 and examine how the point's height, h , changes as the distance traversed, d , changes. Note particularly that each time the point traverses $\frac{1}{8}$ of the circumference of the circle, it travels a distance of $24 \cdot \frac{1}{8} = 3$ units, as seen in Figure 2.1.2 where each noted point lies 3 additional units along the circle beyond the preceding one. Note that we know the exact heights of certain points. Since the circle has circumference $C = 24$, we know that $24 = 2\pi r$ and therefore $r = \frac{12}{\pi} \approx 3.82$. Hence, the point where $d = 6$ (located $1/4$ of the way along the circle) is at a height of $h = \frac{12}{\pi} \approx 3.82$. Doubling this value, the point where $d = 12$ has height $h = \frac{24}{\pi} \approx 7.64$. Other heights, such as those that correspond to $d = 3$ and $d = 15$ (identified on the figure by the green line segments) are not obvious from the circle's radius, but can be estimated from the grid in Figure 2.1.2 as $h \approx 1.1$ (for $d = 3$) and $h \approx 6.5$ (for $d = 15$). Using all of these observations along with the symmetry of the circle, we can determine the other entries in Table 2.1.3. Moreover, if we now let the point continue traversing the circle, we observe that the d -values will increase accordingly, but the h -values will repeat according to the already-established pattern, resulting in the data in Table 2.1.4. It is apparent that each point on the circle corresponds to one and only one height, and thus we can view the

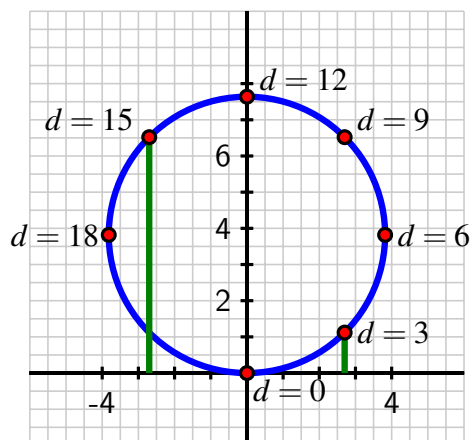


Figure 2.1.2: A point traversing a circle with circumference $C = 24$.

d	0	3	6	9	12	15	18	21	24
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

Table 2.1.3: Data for height, h , as a function of distance traversed, d .

height of a point as a function of the distance the point has traversed around the circle, say $h = f(d)$. Using the data from the two tables and connecting the points in an intuitive way, we get the graph shown in Figure 2.1.5. The function $h = f(d)$ we have been discussing is an example of what we will call a *circular function*. Indeed, it is apparent that if we

- take any circle in the plane,
- choose a starting location for a point on the circle,
- let the point traverse the circle continuously,
- and track the height of the point as it traverses the circle,

the height of the point is a function of distance traversed and the resulting graph will have the same basic shape as the curve shown in Figure 2.1.5. It also turns out that if we track the location of the x -coordinate of the point on the circle, the x -coordinate is also a function of distance traversed and its curve has a similar shape to the graph of the height of the point (the y -coordinate). Both of these functions are circular functions because they are generated by motion around a circle.

Activity 2.1.2. Consider the circle pictured in Figure 2.1.6 that is centered at the point $(2, 2)$ and that has circumference 8. Assume that we track the y -coordinate (that is, the height, h) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.

d	24	27	30	33	36	39	42	45	48
h	0	1.1	3.82	6.5	7.64	6.5	3.82	1.1	0

Table 2.1.4: Additional data for height, h , as a function of distance traversed, d .

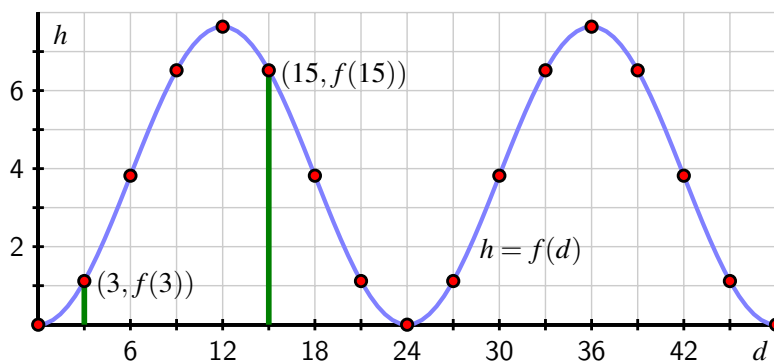


Figure 2.1.5: The height, h , of a point traversing a circle of radius 24 as a function of distance, d , traversed around the circle.

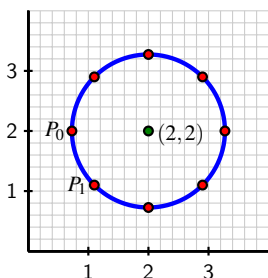


Figure 2.1.6: A point traversing the circle.

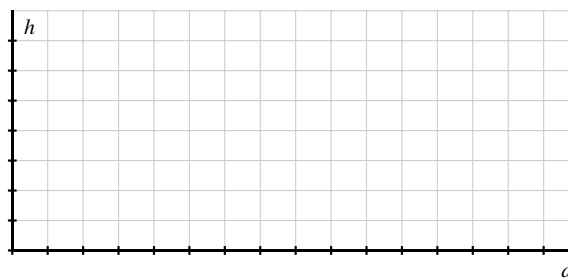


Figure 2.1.7: Axes for plotting h as a function of d .

- How far along the circle is the point P_1 from P_0 ? Why?
- Label the subsequent points in the figure P_2, P_3, \dots as we move counterclockwise around the circle. What is the exact y -coordinate of the point P_2 ? of P_4 ? Why?
- Determine the y -coordinates of the remaining points on the circle (exactly where possible, otherwise approximately) and hence complete the entries in Table 2.1.8 that track the height, h , of the point traversing the circle as a function of distance traveled, d . Note that the d -values in the table correspond to the point traversing the circle more than once.

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h	2																

Table 2.1.8: Data for h as a function of d .

- By plotting the points in Table 2.1.8 and connecting them in an intuitive way, sketch a graph of h as a function of d on the axes provided in Figure 2.1.7 over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.
- What is similar about your graph in comparison to the one in Figure 2.1.5? What is different?
- What will be the value of h when $d = 51$? How about when $d = 102$?

2.1.2 Properties of Circular Functions

Every circular function has several important features that are connected to the circle that defines the function. For the discussion that follows, we focus on circular functions that result from tracking the y -coordinate of a point traversing counterclockwise a circle of radius a centered at the point (k, m) . Further, we will denote the circumference of the circle by the letter p .

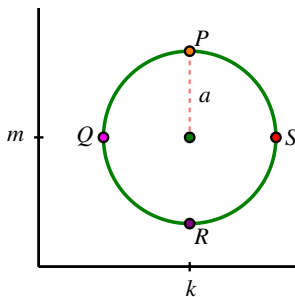


Figure 2.1.9: A point traversing the circle.

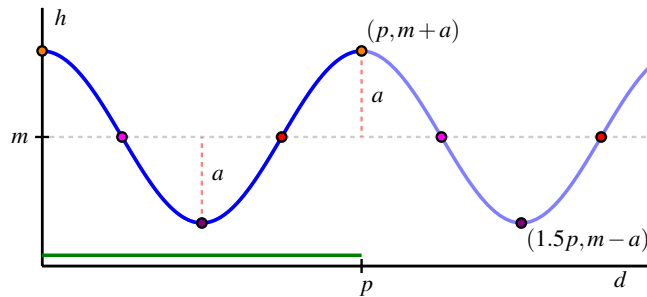


Figure 2.1.10: Plotting h as a function of d .

We assume that the point traversing the circle starts at P in Figure 2.1.9. Its height is initially $y = m + a$, and then its height decreases to $y = m$ as we traverse to Q . Continuing, the point's height falls to $y = m - a$ at R , and then rises back to $y = m$ at S , and eventually back up to $y = m + a$ at the top of the circle. If we plot these heights continuously as a function of distance, d , traversed around the circle, we get the curve shown at right in Figure 2.1.10. This curve has several important features for which we introduce important terminology.

The **midline** of a circular function is the horizontal line $y = m$ for which half the curve lies above the line and half the curve lies below. If the circular function results from tracking the y -coordinate of a point traversing a circle, $y = m$ corresponds to the y -coordinate of

the center of the circle. In addition, the **amplitude** of a circular function is the maximum deviation of the curve from the midline. Note particularly that the value of the amplitude, a , corresponds to the radius of the circle that generates the curve.

Because we can traverse the circle in either direction and for as far as we wish, the domain of any circular function is the set of all real numbers. From our observations about the midline and amplitude, it follows that the range of a circular function with midline $y = m$ and amplitude a is the interval $[m - a, m + a]$.

Finally, we introduce the formal definition of a **periodic** function.

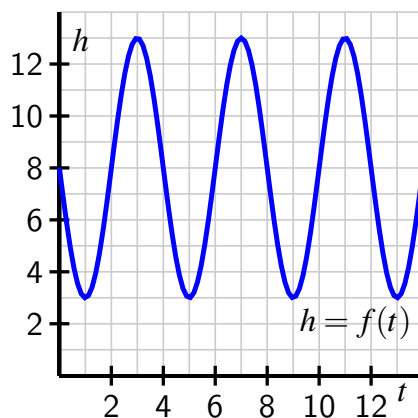
Definition 2.1.11 Let f be a function whose domain and codomain are each the set of all real numbers. We say that f is **periodic** provided that there exists a real number k such that $f(x + k) = f(x)$ for every possible choice of x . The smallest value p for which $f(x + p) = f(x)$ for every choice of x is called the **period** of f . \diamond

For a circular function, the period is always the circumference of the circle that generates the curve. In Figure 2.1.10, we see how the curve has completed one full cycle of behavior every p units, regardless of where we start on the curve.

Circular functions arise as models for important phenomena in the world around us, such as in a *harmonic oscillator*. Consider a mass attached to a spring where the mass sits on a frictionless surface. After setting the mass in motion by stretching or compressing the spring, the mass will oscillate indefinitely back and forth, and its distance from a fixed point on the surface turns out to be given by a circular function.

Activity 2.1.3. A weight is placed on a frictionless table next to a wall and attached to a spring that is fixed to the wall. From its natural position of rest, the weight is imparted an initial velocity that sets it in motion. The weight then oscillates back and forth, and we can measure its distance, $h = f(t)$ (in inches) from the wall at any given time, t (in seconds). A graph of f and a table of select values are given below.

t	$f(t)$	t	$f(t)$
0.25	6.807	2.25	9.913
0.5	4.464	2.5	11.536
0.75	3.381	2.75	12.619
1	3.000	3	13.000
1.25	3.381	3.25	12.619
1.5	4.464	3.5	11.536
1.75	6.087	3.75	9.913
2	8.000	4	8.000



- a. Determine the period p , midline $y = m$, and amplitude a of the function f .

- b. What is the furthest distance the weight is displaced from the wall? What is the least distance the weight is displaced from the wall? What is the range of f ?
- c. Determine the average rate of change of f on the intervals $[4, 4.25]$ and $[4.75, 5]$. Write one careful sentence to explain the meaning of each (including units). In addition, write a sentence to compare the two different values you find and what they together say about the motion of the weight.
- d. Based on the periodicity of the function, what is the value of $f(6.75)$? of $f(11.25)$?

2.1.3 The average rate of change of a circular function

Just as there are important trends in the values of a circular function, there are also interesting patterns in the average rate of change of the function. These patterns are closely tied to the geometry of the circle.

For the next part of our discussion, we consider a circle of radius 1 centered at $(0, 0)$, and consider a point that travels a distance d counterclockwise around the circle with its starting point viewed as $(1, 0)$. We use this circle to generate the circular function $h = f(d)$ that tracks the height of the point at the moment the point has traversed d units around the circle from $(1, 0)$. Let's consider the average rate of change of f on several intervals that are connected to certain fractions of the circumference.

Remembering that h is a function of distance traversed along the circle, it follows that the average rate of change of h on any interval of distance between two points P and Q on the circle is given by

$$AV_{[P,Q]} = \frac{\text{change in height}}{\text{distance along the circle}},$$

where both quantities are measured from point P to point Q .

First, in Figure 2.1.12, we consider points P , Q , and R where Q results from traversing $1/8$ of the circumference from P , and R $1/8$ of the circumference from Q . In particular, we note that the distance d_1 along the circle from P to Q is the same as the distance d_2 along the circle from Q to R , and thus $d_1 = d_2$. At the same time, it is apparent from the geometry of the circle that the change in height h_1 from P to Q is greater than the change in height h_2 from Q to R , so $h_1 > h_2$. Thus, we can say that

$$AV_{[P,Q]} = \frac{h_1}{d_1} > \frac{h_2}{d_2} = AV_{[Q,R]}.$$

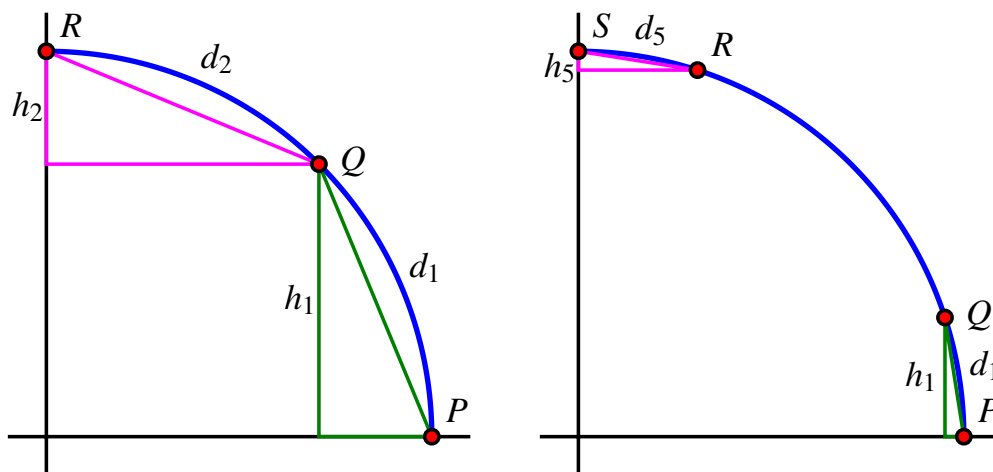


Figure 2.1.12: Comparing the average rate of change over $1/8$ the circumference. **Figure 2.1.13:** Comparing the average rate of change over $1/20$ the circumference.

The differences in certain average rates of change appear to become more extreme if we consider shorter arcs along the circle. Next we consider traveling $1/20$ of the circumference along the circle. In Figure 2.1.13, points P and Q lie $1/20$ of the circumference apart, as do R and S , so here $d_1 = d_5$. In this situation, it is the case that $h_1 > h_5$ for the same reasons as above, but we can say even more. From the green triangle in Figure 2.1.13, we see that $h_1 \approx d_1$ (while $h_1 < d_1$), so that $AV_{[P,Q]} = \frac{h_1}{d_1} \approx 1$. At the same time, in the magenta triangle in the figure we see that h_5 is very small, especially in comparison to d_5 , and thus $AV_{[R,S]} = \frac{h_5}{d_5} \approx 0$. Hence, in Figure 2.1.13,

$$AV_{[P,Q]} \approx 1 \text{ and } AV_{[R,S]} \approx 0.$$

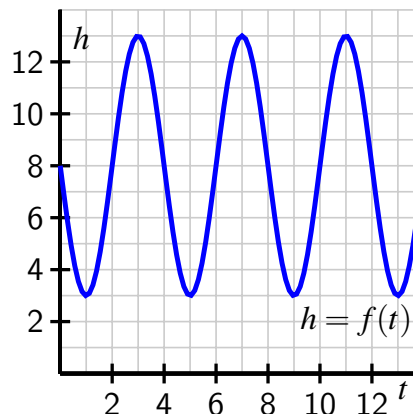
This information tells us that a circular function appears to change most rapidly for points near its midline and to change least rapidly for points near its highest and lowest values.

We can study the average rate of change not only on the circle itself, but also on a graph such as Figure 2.1.10, and thus make conclusions about where the function is increasing, decreasing, concave up, and concave down.

Activity 2.1.4. Consider the same setting as Activity 2.1.3: a weight oscillates back and forth on a frictionless table with distance from the wall given by, $h = f(t)$ (in inches) at any given time, t (in seconds). A graph of f and a table of select values are given below.

t	$f(t)$
0.25	6.807
0.5	4.464
0.75	3.381
1	3.000
1.25	3.381
1.5	4.464
1.75	6.087
2	8.000

t	$f(t)$
2.25	9.913
2.5	11.536
2.75	12.619
3	13.000
3.25	12.619
3.5	11.536
3.75	9.913
4	8.000



- Determine $AV_{[2,2.25]}$, $AV_{[2.25,2.5]}$, $AV_{[2.5,2.75]}$, and $AV_{[2.75,3]}$. What do these four values tell us about how the weight is moving on the interval $[2, 3]$?
- Give an example of an interval of length 0.25 units on which f has its most negative average rate of change. Justify your choice.
- Give an example of the longest interval you can find on which f is decreasing.
- Give an example of an interval on which f is concave up.²
- On an interval where f is both decreasing and concave down, what does this tell us about how the weight is moving on that interval? For instance, is the weight moving toward or away from the wall? is it speeding up or slowing down?
- What general conclusions can you make about the average rate of change of a circular function on intervals near its highest or lowest points? about its average rate of change on intervals near the function's midline?

2.1.4 Summary

- When a point traverses a circle, a corresponding function can be generated by tracking the height of the point as it moves around the circle, where height is viewed as a function of distance traveled around the circle. We call such a function a *circular function*. An image that shows how a circular function's graph is generated from the circle can be seen in Figure 2.1.10.
- Circular functions have several standard features. The function has a *midline* that is the line for which half the points on the curve lie above the line and half the points on the curve lie below. A circular function's *amplitude* is the maximum deviation of the

²Recall that a function is concave up on an interval provided that throughout the interval, the curve bends upward, similar to a parabola that opens up.

function value from the midline; the amplitude corresponds to the radius of the circle that generates the function. Circular functions also repeat themselves, and we call the smallest value of p for which $f(x + p) = f(x)$ for all x the period of the function. The period of a circular function corresponds to the circumference of the circle that generates the function.

- Non-constant linear functions are either always increasing or always decreasing; quadratic functions are either always concave up or always concave down. Circular functions are sometimes increasing and sometimes decreasing, plus sometimes concave up and sometimes concave down. These behaviors are closely tied to the geometry of the circle.

2.1.5 Exercises



1. Let $y = f(x)$ be a periodic function whose values are given below. Find the period, amplitude, and midline.

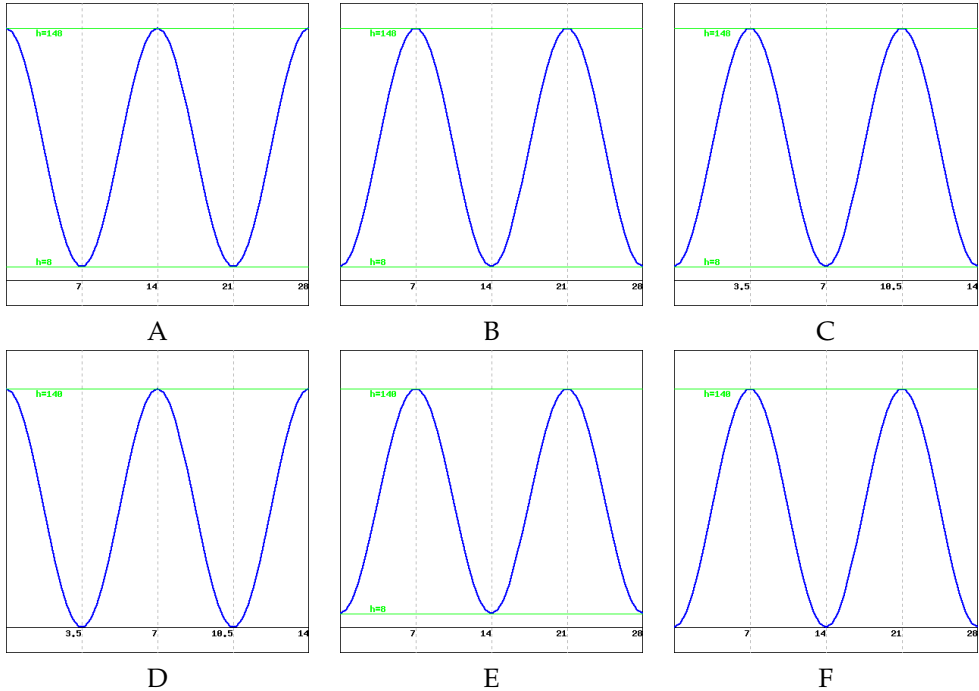
x	5	25	45	65	85	105	125	145	165
f(x)	17	15	-3	17	15	-3	17	15	-3



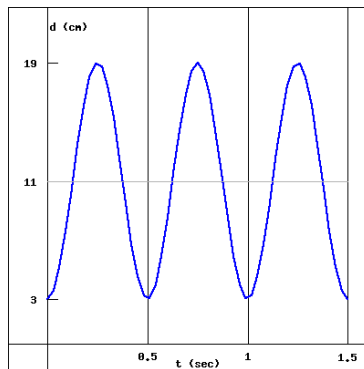
2. A ferris wheel is 140 meters in diameter and boarded at its lowest point (6 O'Clock) from a platform which is 8 meters above ground. The wheel makes one full rotation every 14 minutes, and at time $t = 0$ you are at the loading platform (6 O'Clock). Let $h = f(t)$ denote your height above ground in meters after t minutes.

- What is the period of the function $h = f(t)$?
- What is the midline of the function $h = f(t)$?
- What is the amplitude of the function $h = f(t)$?
- Consider the six possible graphs of $h = f(t)$ below. Be sure to carefully read the labels on the axes in order distinguish the key features of each graph.

Which (if any) of the graphs A-F represents two full revolutions of the ferris wheel described above?



3. A weight is suspended from the ceiling by a spring. Let d be the distance in centimeters from the ceiling to the weight. When the weight is motionless, $d = 11$ cm. If the weight is disturbed, it begins to bob up and down, or *oscillate*. Then d is a periodic function of t , the time in seconds, so $d = f(t)$. Consider the graph of $d = f(t)$ below, which represents the distance of the weight from the ceiling at time t .



(a) Based on the graph of $d = f(t)$ above, which of the statements below correctly describes the motion of the weight as it bobs up and down?

- The weight starts closest to the floor and begins by bouncing up towards the ceiling.
- The weight starts closest to the ceiling and begins by stretching the spring down towards the floor.
- The spring starts at its average distance between the ceiling and floor and begins

by stretching the spring down towards the floor.

Ⓒ None of the above

(b) How long does it take the weight to bounce completely up and down (or down and up) and return to its starting position?

(c) What is the closest the weight gets to the ceiling?

(d) What is the furthest the weights gets from the ceiling?

(e) What is the amplitude of the graph of $d = f(t)$?



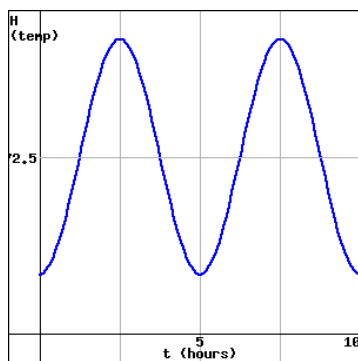
4. The temperature of a chemical reaction oscillates between a low of 10°C and a high of 135°C . The temperature is at its lowest point at time $t = 0$, and reaches its maximum point over a two and a half hour period. It then takes the same amount of time to return back to its initial temperature. Let $y = H(t)$ denote the temperature of the reaction t hours after the reaction begins.

(a) What is the period of the function $y = H(t)$?

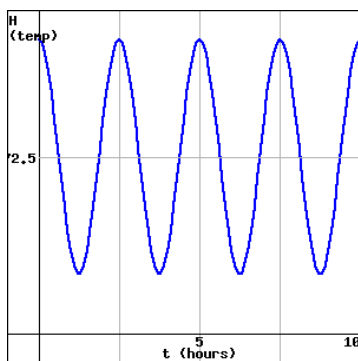
(b) What is the midline of the function $y = H(t)$?

(c) What is the amplitude of the function $y = H(t)$?

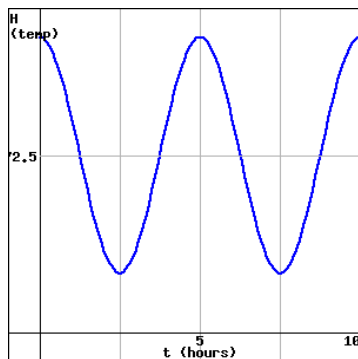
(d) Based on your answers above, make a graph of the function $y = H(t)$ on a piece of paper. Which of the graphs below best matches your graph?



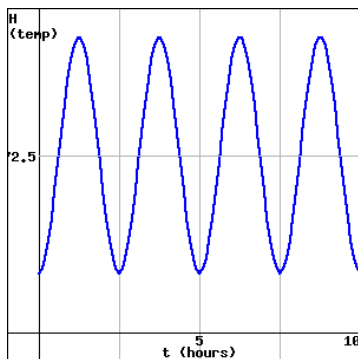
A



B



C



D

5. Consider the circle pictured in Figure 2.1.14 that is centered at the point $(2, 2)$ and that has circumference 8. Suppose that we track the x -coordinate (that is, the horizontal location, which we will call k) of a point that is traversing the circle counterclockwise and that it starts at P_0 as pictured.

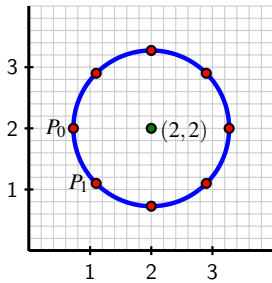


Figure 2.1.14: A point traversing the circle.

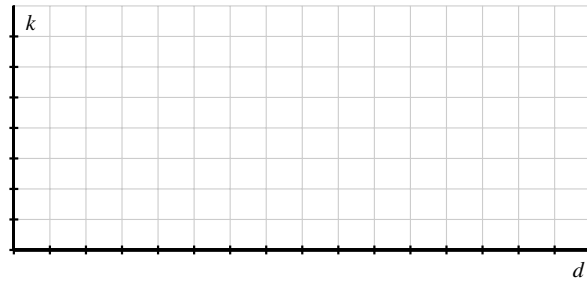


Figure 2.1.15: Axes for plotting k as a function of d .

Recall that in Activity 2.1.2 we identified the exact and approximate vertical coordinates of all 8 noted points on the unit circle. In addition, recall that the radius of the circle is $r = \frac{8}{2\pi} \approx 1.2732$.

- What is the exact horizontal coordinate of P_0 ? Why?
- Complete the entries in Table 2.1.16 that track the horizontal location, k , of the point traversing the circle as a function of distance traveled, d .

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k	0.73																

Table 2.1.16: Data for h as a function of d .

- By plotting the points in Table 2.1.16 and connecting them in an intuitive way, sketch a graph of k as a function of d on the axes provided in Figure 2.1.15 over the interval $0 \leq d \leq 16$. Clearly label the scale of your axes and the coordinates of several important points on the curve.
 - What is similar about your graph in comparison to the one in Figure 2.1.7? What is different?
 - What will be the value of k when $d = 51$? How about when $d = 102$?
6. Two circular functions, f and g , are generated by tracking the y -coordinate of a point traversing two different circles. The resulting graphs are shown in Figure 2.1.17 and Figure 2.1.18. Assuming the horizontal scale matches the vertical scale, answer the following questions for each of the functions f and g .

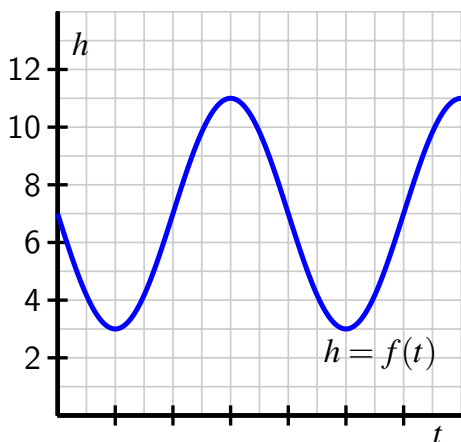


Figure 2.1.17: A plot of the circular function f .

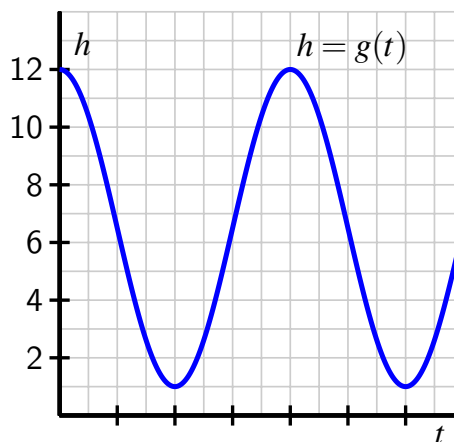


Figure 2.1.18: A plot of the circular function g .

- a. Assume that the circle used to generate the circular function is centered at the point $(0, m)$ and has radius r . What are the numerical values of m and r ? Why?
 - b. What are the coordinates of the location on the circle at which the point begins its traverse? Said differently, what point on the circle corresponds to $t = 0$ on the function's graph?
 - c. What is the period of the function? How is this connected to the circle and to the scale on the horizontal axes on which the function is graphed?
 - d. How would the graph look if the circle's radius was 1 unit larger? 1 unit smaller?
7. A person goes for a ride on a ferris wheel. They enter one of the cars at the lowest possible point on the wheel from a platform 7 feet off the ground. When they are at the very top of the wheel, they are 92 feet off the ground. Let h represent the height of the car (in feet) and d (in feet) the distance the car has traveled along the wheel's circumference from its starting location at the bottom of the wheel. We'll use the notation $h = f(d)$ for how height is a function of distance traveled.
- a. How high above the ground is the center of the ferris wheel?
 - b. How far does the car travel in one complete trip around the wheel?
 - c. For the circular function $h = f(d)$, what is its amplitude? midline? period?
 - d. Sketch an accurate graph of h through at least two full periods. Clearly label the scale on the horizontal and vertical axes along with several important points.

5.3 right triangles

5.3.1 right triangles/ratios (Active Prelude to Calculus)

5.3.2 using inverse trig functions to find angles in triangles (Active Prelude to Calculus)

5.3.3 defining trig functions with right triangles, special right triangles, using trig functions to find side length and in applications (OpenStax Algebra and Trigonometry)

5.3.4 defining trig functions with unit circle (OpenStax Algebra and Trigonometry)

Trigonometry

4.1 Right triangles

Motivating Questions

- How can we view $\cos(\theta)$ and $\sin(\theta)$ as side lengths in right triangles with hypotenuse 1?
 - Why can both $\cos(\theta)$ and $\sin(\theta)$ be thought of as ratios of certain side lengths in any right triangle?
 - What is the minimum amount of information we need about a right triangle in order to completely determine all of its sides and angles?
-

In Section 2.3, we defined the cosine and sine functions as the functions that track the location of a point traversing the unit circle counterclockwise from $(1, 0)$. In particular, for a central angle of radian measure t that passes through the point $(1, 0)$, we define $\cos(t)$ as the x -coordinate of the point where the other side of the angle intersects the unit circle, and $\sin(t)$ as the y -coordinate of that same point, as pictured in Figure 4.1.1.

By changing our perspective slightly, we can see that it is equivalent to think of the values of the sine and cosine function as representing the lengths of legs in right triangles. Specifically, given a central angle¹ θ , if we think of the right triangle with vertices $(\cos(\theta), 0)$, $(0, 0)$, and $(\cos(\theta), \sin(\theta))$, then the length of the horizontal leg is $\cos(\theta)$ and the length of the vertical leg is $\sin(\theta)$, as seen in Figure 4.1.2.

¹In our work with right triangles, we'll often represent the angle by θ and think of this angle as fixed, as opposed to our previous use of t where we frequently think of t as changing.

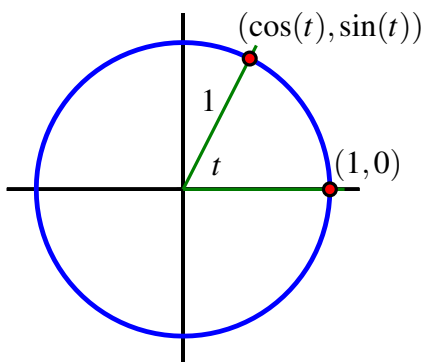


Figure 4.1.1: The values of $\cos(t)$ and $\sin(t)$ as coordinates on the unit circle.

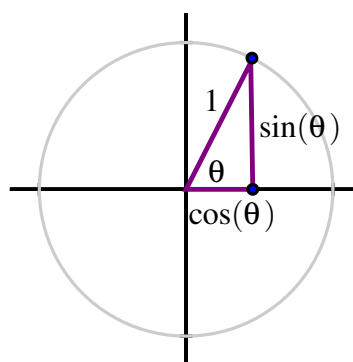


Figure 4.1.2: The values of $\cos(\theta)$ and $\sin(\theta)$ as the lengths of the legs of a right triangle.

This right triangle perspective enables us to use the sine and cosine functions to determine missing information in certain right triangles. The field of mathematics that studies relationships among the angles and sides of triangles is called *trigonometry*. In addition, it's important to recall both the Pythagorean Theorem and the Fundamental Trigonometric Identity.

The former states that in any right triangle with legs of length a and b and hypotenuse of length c , it follows $a^2 + b^2 = c^2$. The latter, which is a special case of the Pythagorean Theorem, says that for any angle θ , $\cos^2(\theta) + \sin^2(\theta) = 1$.

Preview Activity 4.1.1. For each of the following situations, sketch a right triangle that satisfies the given conditions, and then either determine the requested missing information in the triangle or explain why you don't have enough information to determine it. Assume that all angles are being considered in radian measure.

- The length of the other leg of a right triangle with hypotenuse of length 1 and one leg of length $\frac{3}{5}$.
- The lengths of the two legs in a right triangle with hypotenuse of length 1 where one of the non-right angles measures $\frac{\pi}{3}$.
- The length of the other leg of a right triangle with hypotenuse of length 7 and one leg of length 6.
- The lengths of the two legs in a right triangle with hypotenuse 5 and where one of the non-right angles measures $\frac{\pi}{4}$.
- The length of the other leg of a right triangle with hypotenuse of length 1 and one leg of length $\cos(0.7)$.
- The measures of the two angles in a right triangle with hypotenuse of length 1 where the two legs have lengths $\cos(1.1)$ and $\sin(1.1)$, respectively.

4.1.1 The geometry of triangles

In the study of functions, linear functions are the simplest of all and form a foundation for our understanding of functions that have other shapes. In the study of geometric shapes (polygons, circles, and more), the simplest figure of all is the triangle, and understanding triangles is foundational to understanding many other geometric ideas. To begin, we list some familiar and important facts about triangles.

- Any triangle has 6 important features: 3 sides and 3 angles.
- In any triangle in the Cartesian plane, the sum of the measures of the interior angles is π radians (or equivalently, 180°).
- In any triangle in the plane, knowing three of the six features of a triangle is often enough information to determine the missing three features.²

The situation is especially nice for right triangles, because then we only have five unknown features since one of the angles is $\frac{\pi}{2}$ radians (or 90°), as demonstrated in Figure 4.1.3. If we know one of the two non-right angles, then we know the other as well. Moreover, if we know any two sides, we can immediately deduce the third, because of the Pythagorean Theorem. As we saw in Preview Activity 4.1.1, the cosine and sine functions offer additional help in determining missing information in right triangles. Indeed, while the functions $\cos(t)$ and $\sin(t)$ have many important applications in modeling periodic phenomena such as oscillating masses on springs, they also find powerful application in settings involving right triangles, such as in navigation and surveying.

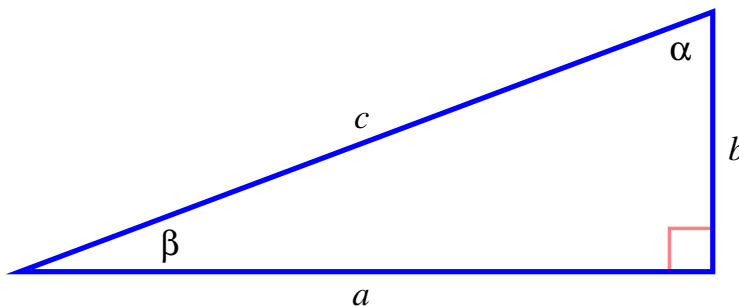


Figure 4.1.3: The 5 potential unknowns in a right triangle.

Because we know the values of the cosine and sine functions from the unit circle, right triangles with hypotenuse 1 are the easiest ones in which to determine missing information. In addition, we can relate any other right triangle to a right triangle with hypotenuse 1 through the concept of *similarity*. Recall that two triangles are **similar** provided that one is a magnification of the other. More precisely, two triangles are similar whenever there is some

²Formally, this idea relies on what are called *congruence criteria*. For instance, if we know the lengths of all three sides, then the angle measures of the triangle are uniquely determined. This is called the Side-Side-Side Criterion (SSS). You are likely familiar with SSS, as well as SAS (Side-Angle-Side), ASA, and AAS, which are the four standard criteria.

constant k such that every side in one triangle is k times as long as the corresponding side in the other and the corresponding angles in the two triangles are equal. An important result from geometry tells us that if two triangles are known to have all three of their corresponding angles equal, then it follows that the two triangles are similar, and therefore their corresponding sides must be proportionate to one another.

Activity 4.1.2. Consider right triangle OPQ given in Figure 4.1.4, and assume that the length of the hypotenuse is $OP = r$ for some constant $r > 1$. Let point M lie on \overline{OP} between O and P in such a way that $OM = 1$, and let point N lie on \overline{OQ} so that $\angle ONM$ is a right angle, as pictured. In addition, assume that point O corresponds to $(0, 0)$, point Q to $(x, 0)$, and point P to (x, y) so that $OQ = x$ and $PQ = y$. Finally, let θ be the measure of $\angle POQ$.

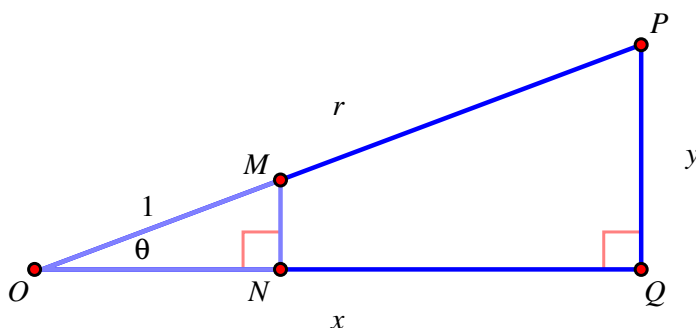


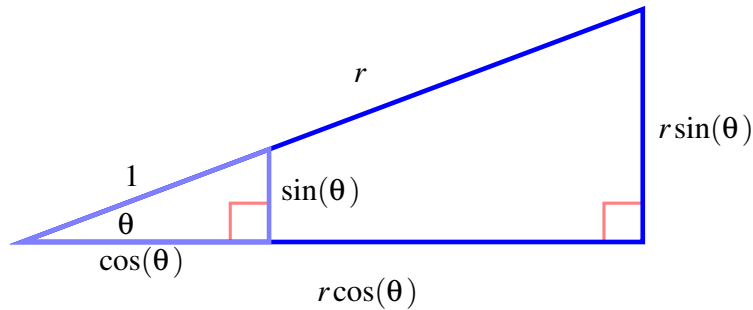
Figure 4.1.4: Two right triangles $\triangle OPQ$ and $\triangle OMN$.

- Explain why $\triangle OPQ$ and $\triangle OMN$ are similar triangles.
- What is the value of the ratio $\frac{OP}{OM}$? What does this tell you about the ratios $\frac{OQ}{ON}$ and $\frac{PQ}{MN}$?
- What is the value of ON in terms of θ ? What is the value of MN in terms of θ ?
- Use your conclusions in (b) and (c) to express the values of x and y in terms of r and θ .

4.1.2 Ratios of sides in right triangles

A right triangle with a hypotenuse of length 1 can be viewed as lying in standard position in the unit circle, with one vertex at the origin and one leg along the positive x -axis. If we let the angle formed by the hypotenuse and the horizontal leg have measure θ , then the right triangle with hypotenuse 1 has horizontal leg of length $\cos(\theta)$ and vertical leg of length $\sin(\theta)$. If we consider now consider a similar right triangle with hypotenuse of length $r \neq 1$, we can view that triangle as a magnification of a triangle with hypotenuse 1. These observations, combined with our work in Activity 4.1.2, show us that the horizontal legs of the right triangle with hypotenuse r have measure $r \cos(\theta)$ and $r \sin(\theta)$, as pictured in

Figure 4.1.5.

**Figure 4.1.5:** The roles of r and θ in a right triangle.

From the similar triangles in Figure 4.1.5, we can make an important observation about ratios in right triangles. Because the triangles are similar, the ratios of corresponding sides must be equal, so if we consider the two hypotenuses and the two horizontal legs, we have

$$\frac{r}{1} = \frac{r \cos(\theta)}{\cos(\theta)}. \quad (4.1.1)$$

If we rearrange Equation (4.1.1) by dividing both sides by r and multiplying both sides by $\cos(\theta)$, we see that

$$\frac{\cos(\theta)}{1} = \frac{r \cos(\theta)}{r}. \quad (4.1.2)$$

From a geometric perspective, Equation (4.1.2) tells us that the ratio of the horizontal leg of a right triangle to the hypotenuse of the triangle is always the same (regardless of r) and that the value of that ratio is $\cos(\theta)$, where θ is the angle adjacent to the horizontal leg. In an analogous way, the equation involving the hypotenuses and vertical legs of the similar triangles is

$$\frac{r}{1} = \frac{r \sin(\theta)}{\sin(\theta)}, \quad (4.1.3)$$

which can be rearranged to

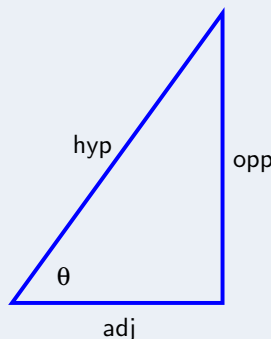
$$\frac{\sin(\theta)}{1} = \frac{r \sin(\theta)}{r}. \quad (4.1.4)$$

Equation (4.1.4) shows that the ratio of the vertical leg of a right triangle to the hypotenuse of the triangle is always the same (regardless of r) and that the value of that ratio is $\sin(\theta)$, where θ is the angle opposite the vertical leg. We summarize these recent observations as follows.

Ratios in right triangles.

In a right triangle where one of the non-right angles is θ , and “adj” denotes the length of the leg adjacent to θ , “opp” the length the side opposite θ , and “hyp” the length of the hypotenuse,

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \text{ and } \sin(\theta) = \frac{\text{opp}}{\text{hyp}}.$$



Activity 4.1.3. In each of the following scenarios involving a right triangle, determine the exact values of as many of the remaining side lengths and angle measures (in radians) that you can. If there are quantities that you cannot determine, explain why. For every prompt, draw a labeled diagram of the situation.

- A right triangle with hypotenuse 7 and one non-right angle of measure $\frac{\pi}{7}$.
- A right triangle with non-right angle α that satisfies $\sin(\alpha) = \frac{3}{5}$.
- A right triangle where one of the non-right angles is $\beta = 1.2$ and the hypotenuse has length 2.7.
- A right triangle with hypotenuse 13 and one leg of length 6.5.
- A right triangle with legs of length 5 and 12.
- A right triangle where one of the non-right angles is $\beta = \frac{\pi}{5}$ and the leg opposite this angle has length 4.

4.1.3 Using a ratio involving sine and cosine

In Activity 4.1.3, we found that in many cases where we have a right triangle, knowing two additional pieces of information enables us to find the remaining three unknown quantities in the triangle. At this point in our studies, the following general principles hold.

Missing information in right triangles.

In any right triangle,

- if we know one of the non-right angles and the length of the hypotenuse, we can find both the remaining non-right angle and the lengths of the two legs;
- if we know the length of two sides of the triangle, then we can find the length

of the other side;

- if we know the measure of one non-right angle, then we can find the measure of the remaining angle.

In scenario (1.), all 6 features of the triangle are not only determined, but we are able to find their values. In (2.), the triangle is uniquely determined by the given information, but as in Activity 4.1.3 parts (d) and (e), while we know the values of the sine and cosine of the angles in the triangle, we haven't yet developed a way to determine the measures of those angles. Finally, in scenario (3.), the triangle is not uniquely determined, since any magnified version of the triangle will have the same three angles as the given one, and thus we need more information to determine side length.

We will revisit scenario (2) in our future work. Now, however, we want to consider a situation that is similar to (1), but where it is one leg of the triangle instead of the hypotenuse that is known. We encountered this in Activity 4.1.3 part (f): a right triangle where one of the non-right angles is $\beta = \frac{\pi}{5}$ and the leg opposite this angle has length 4.

Example 4.1.6

Consider a right triangle in which one of the non-right angles is $\beta = \frac{\pi}{5}$ and the leg opposite β has length 4.

Determine (both exactly and approximately) the measures of all of the remaining sides and angles in the triangle.

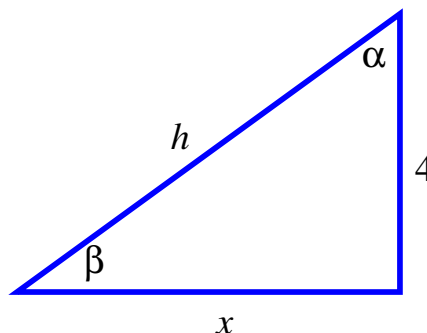


Figure 4.1.7: The given right triangle.

Solution. From the fact that $\beta = \frac{\pi}{5}$, it follows that $\alpha = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$. In addition, we know that

$$\sin\left(\frac{\pi}{5}\right) = \frac{4}{h} \quad (4.1.5)$$

and

$$\cos\left(\frac{\pi}{5}\right) = \frac{x}{h} \quad (4.1.6)$$

Solving Equation (4.1.5) for h , we see that

$$h = \frac{4}{\sin\left(\frac{\pi}{5}\right)}, \quad (4.1.7)$$

which is the exact numerical value of h . Substituting this result in Equation (4.1.6), solving for h we find that

$$\cos\left(\frac{\pi}{5}\right) = \frac{x}{\frac{4}{\sin\left(\frac{\pi}{5}\right)}}. \quad (4.1.8)$$

Solving this equation for the single unknown x shows that

$$x = \frac{4 \cos\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)}.$$

The approximate values of x and h are $x \approx 5.506$ and $h \approx 6.805$. □

Example 4.1.6 demonstrates that a ratio of values of the sine and cosine function can be needed in order to determine the value of one of the missing sides of a right triangle, and also that we may need to work with two unknown quantities simultaneously in order to determine both of their values.

Activity 4.1.4.

We want to determine the distance between two points A and B that are directly across from one another on opposite sides of a river, as pictured in Figure 4.1.8. We mark the locations of those points and walk 50 meters downstream from B to point P and use a sextant to measure $\angle BPA$. If the measure of $\angle BPA$ is 56.4° , how wide is the river? What other information about the situation can you determine?

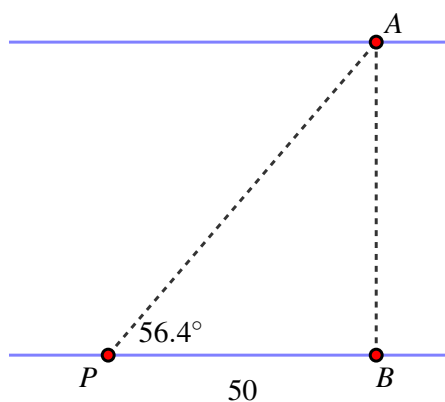


Figure 4.1.8: Finding the width of the river.

4.1.4 Summary

- In a right triangle with hypotenuse 1, we can view $\cos(\theta)$ as the length of the leg adjacent to θ and $\sin(\theta)$ as the length of the leg opposite θ , as seen in Figure 4.1.2. This is simply a change in perspective achieved by focusing on the triangle as opposed to the unit circle.
- Because a right triangle with hypotenuse of length r can be thought of as a scaled version of a right triangle with hypotenuse of length 1, we can conclude that in a right triangle with hypotenuse of length r , the leg adjacent to angle θ has length $r \cos(\theta)$, and the leg opposite θ has length $r \sin(\theta)$, as seen in Figure 4.1.5. Moreover, in any right triangle with angle θ , we know that

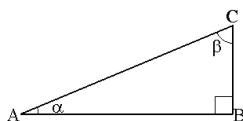
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \text{ and } \sin(\theta) = \frac{\text{opp}}{\text{hyp}}.$$

- In a right triangle, there are five additional characteristics: the measures of the two

non-right angles and the lengths of the three sides. In general, if we know one of those two angles and one of the three sides, we can determine all of the remaining pieces.

4.1.5 Exercises

1. Refer to the right triangle in the figure.



If $BC = 3$ and the angle $\alpha = 65^\circ$, find any missing angles or sides.

2. Suppose that a , b and c are the sides of a right triangle, where side a is across from angle A , side b is across from angle B , and side c is across from the right angle. If $a = 17$ and $B = 33^\circ$, find the missing sides and angles in this right triangle. All angles should be in degrees (not radians), and all trig functions entered will be evaluated in degrees (not radians).
3. A person standing 50 feet away from a streetlight observes that they cast a shadow that is 14 feet long. If a ray of light from the streetlight to the tip of the person's shadow forms an angle of 27.5° with the ground, how tall is the person and how tall is the streetlight? What other information about the situation can you determine?
4. A person watching a rocket launch uses a laser range-finder to measure the distance from themselves to the rocket. The range-finder also reports the angle at which the finder is being elevated from horizontal. At a certain instant, the range-finder reports that it is elevated at an angle of 17.4° from horizontal and that the distance to the rocket is 1650 meters. How high off the ground is the rocket? Assuming a straight-line vertical path for the rocket that is perpendicular to the earth, how far away was the rocket from the range-finder at the moment it was launched?
5. A trough is constructed by bending a $4' \times 24'$ rectangular sheet of metal. Two symmetric folds 2 feet apart are made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured in Figure 4.1.9. Determine a formula for $V(\theta)$, the volume of the trough as a function of θ .

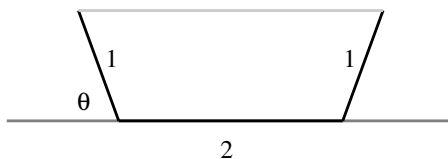


Figure 4.1.9: A cross-section of the trough.

Hint. The volume of the trough is the area of a cross-section times the length of the trough.

4.4 Finding Angles

Motivating Questions

- How can we use inverse trigonometric functions to determine missing angles in right triangles?
 - What situations require us to use technology to evaluate inverse trigonometric functions?
-

In our earlier work in Section 4.1 and Section 4.2, we observed that in any right triangle, if we know the measure of one additional angle and the length of one additional side, we can determine all of the other parts of the triangle. With the inverse trigonometric functions that we developed in Section 4.3, we are now also able to determine the missing angles in any right triangle where we know the lengths of two sides.

While the original trigonometric functions take a particular angle as input and provide an output that can be viewed as the ratio of two sides of a right triangle, the inverse trigonometric functions take an input that can be viewed as a ratio of two sides of a right triangle and produce the corresponding angle as output. Indeed, it's imperative to remember that statements such as

$$\arccos(x) = \theta \quad \text{and} \quad \cos(\theta) = x$$

say the exact same thing from two different perspectives, and that we read “ $\arccos(x)$ ” as “the angle whose cosine is x ”.

Preview Activity 4.4.1. Consider a right triangle that has one leg of length 3 and another leg of length $\sqrt{3}$. Let θ be the angle that lies opposite the shorter leg.

- Sketch a labeled picture of the triangle.
- What is the exact length of the triangle's hypotenuse?
- What is the exact value of $\sin(\theta)$?
- Rewrite your equation from (b) using the arcsine function in the form $\arcsin(\square) = \Delta$, where \square and Δ are numerical values.
- What special angle from the unit circle is θ ?

4.4.1 Evaluating inverse trigonometric functions

Like the trigonometric functions themselves, there are a handful of important values of the inverse trigonometric functions that we can determine exactly without the aid of a computer. For instance, we know from the unit circle (Figure 2.3.1) that $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$, $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$, and $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$. In these evaluations, we have to be careful to remember that the range of the arccosine function is $[0, \pi]$, while the range of the arcsine

function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and the range of the arctangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, in order to ensure that we choose the appropriate angle that results from the inverse trigonometric function.

In addition, there are many other values at which we may wish to know the angle that results from an inverse trigonometric function. To determine such values, we use a computational device (such as *Desmos*) in order to evaluate the function.

Example 4.4.1

Consider the right triangle pictured in Figure 4.4.2 and assume we know that the vertical leg has length 1 and the hypotenuse has length 3. Let α be the angle opposite the known leg. Determine exact and approximate values for all of the remaining parts of the triangle.

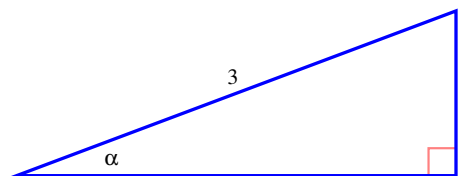


Figure 4.4.2: A right triangle with one known leg and known hypotenuse.

Solution. Because we know the hypotenuse and the side opposite α , we observe that $\sin(\alpha) = \frac{1}{3}$. Rewriting this statement using inverse function notation, we have equivalently that $\alpha = \arcsin(\frac{1}{3})$, which is the exact value of α . Since this is not one of the known special angles on the unit circle, we can find a numerical estimate of α using a computational device. Entering $\arcsin(1/3)$ in *Desmos*, we find that $\alpha \approx 0.3398$ radians. Note well: whatever device we use, we need to be careful to use degree or radian mode as dictated by the problem we are solving. We will always work in radians unless stated otherwise.

We can now find the remaining leg's length and the remaining angle's measure. If we let x represent the length of the horizontal leg, by the Pythagorean Theorem we know that

$$x^2 + 1^2 = 3^2,$$

and thus $x^2 = 8$ so $x = \sqrt{8} \approx 2.8284$. Calling the remaining angle β , since $\alpha + \beta = \frac{\pi}{2}$, it follows that

$$\beta = \frac{\pi}{2} - \arcsin\left(\frac{1}{3}\right) \approx 1.2310.$$

□

Activity 4.4.2. For each of the following different scenarios, draw a picture of the situation and use inverse trigonometric functions appropriately to determine the missing information both exactly and approximately.

- Consider a right triangle with legs of length 11 and 13. What are the measures (in radians) of the non-right angles and what is the length of the hypotenuse?
- Consider an angle α in standard position (vertex at the origin, one side on the positive x -axis) for which we know $\cos(\alpha) = -\frac{1}{2}$ and α lies in quadrant III. What is the measure of α in radians? In addition, what is the value of $\sin(\alpha)$?
- Consider an angle β in standard position for which we know $\sin(\beta) = 0.1$ and β lies in quadrant II. What is the measure of β in radians? In addition, what is the value of $\cos(\beta)$?

4.4.2 Finding angles in applied contexts

Now that we have developed the (restricted) sine, cosine, and tangent functions and their respective inverses, in any setting in which we have a right triangle together with one side length and any one additional piece of information (another side length or a non-right angle measurement), we can determine all of the remaining pieces of the triangle. In the activities that follow, we explore these possibilities in a variety of different applied contexts.

Activity 4.4.3. A roof is being built with a “7-12 pitch.” This means that the roof rises 7 inches vertically for every 12 inches of horizontal span; in other words, the slope of the roof is $\frac{7}{12}$. What is the exact measure (in degrees) of the angle the roof makes with the horizontal? What is the approximate measure? What are the exact and approximate measures of the angle at the peak of the roof (made by the front and back portions of the roof that meet to form the ridge)?

Activity 4.4.4. On a baseball diamond (which is a square with 90-foot sides), the third baseman fields the ball right on the line from third base to home plate and 10 feet away from third base (towards home plate). When he throws the ball to first base, what angle (in degrees) does the line the ball travels make with the first base line? What angle does it make with the third base line? Draw a well-labeled diagram to support your thinking.

What angles arise if he throws the ball to second base instead?

Activity 4.4.5. A camera is tracking the launch of a SpaceX rocket. The camera is located 4000' from the rocket's launching pad, and the camera elevates in order to keep the rocket in focus. At what angle θ (in radians) is the camera tilted when the rocket is 3000' off the ground? Answer both exactly and approximately.

Now, rather than considering the rocket at a fixed height of 3000', let its height vary and call the rocket's height h . Determine the camera's angle, θ as a function of h , and compute the average rate of change of θ on the intervals $[3000, 3500]$, $[5000, 5500]$, and $[7000, 7500]$. What do you observe about how the camera angle is changing?

4.4.3 Summary

- Anytime we know two side lengths in a right triangle, we can use one of the inverse trigonometric functions to determine the measure of one of the non-right angles. For instance, if we know the values of opp and adj in Figure 4.4.3, then since

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}},$$

it follows that $\theta = \arctan\left(\frac{\text{opp}}{\text{adj}}\right)$.

If we instead know the hypotenuse and one of the two legs, we can use either the arcsine or arccosine function accordingly.

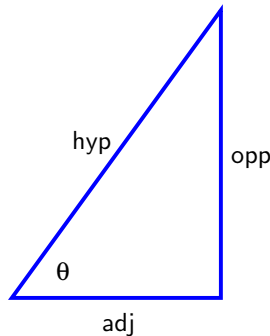





Figure 4.4.3: Finding an angle from knowing the legs in a right triangle.

- For situations other than angles or ratios that involve the 16 special points on the unit circle, technology is required in order to evaluate inverse trigonometric functions. For instance, from the unit circle we know that $\arccos(\frac{1}{2}) = \frac{\pi}{3}$ (exactly), but if we want to know $\arccos(\frac{1}{3})$, we have to estimate this value using a computational device such as *Desmos*. We note that “ $\arccos(\frac{1}{3})$ ” is the exact value of the angle whose cosine is $\frac{1}{3}$.

4.4.4 Exercises

1. If $\cos(\phi) = 0.7087$ and $3\pi/2 \leq \phi \leq 2\pi$, approximate the following to four decimal places. 
 - (a) $\sin(\phi)$
 - (b) $\tan(\phi)$
2. Suppose $\sin \theta = \frac{x}{7}$ and the angle θ is in the first quadrant. Write algebraic expressions for $\cos(\theta)$ and $\tan(\theta)$ in terms of x . 
 - (a) $\cos(\theta)$
 - (b) $\tan(\theta)$
3. Using inverse trigonometric functions, find a solution to the equation $\cos(x) = 0.7$ in the interval $0 \leq x \leq 4\pi$. Then, use a graph to find all other solutions to this equation in this interval. Enter your answers as a comma separated list. 
4. At an airshow, a pilot is flying low over a runway while maintaining a constant altitude of 2000 feet and a constant speed. On a straight path over the runway, the pilot observes on her laser range-finder that the distance from the plane to a fixed building adjacent to the runway is 7500 feet. Five seconds later, she observes that distance to the same building is now 6000 feet.
 - a. What is the angle of depression from the plane to the building when the plane is 7500 feet away from the building? (The angle of depression is the angle that the pilot’s line of sight makes with the horizontal.)
 - b. What is the angle of depression when the plane is 6000 feet from the building?

- c. How far did the plane travel during the time between the two different observations?
 - d. What is the plane's velocity (in miles per hour)?
5. On a calm day, a photographer is filming a hot air balloon. When the balloon launches, the photographer is stationed 850 feet away from the balloon.
 - a. When the balloon is 200 feet off the ground, what is the angle of elevation of the camera?
 - b. When the balloon is 275 feet off the ground, what is the angle of elevation of the camera?
 - c. Let θ represent the camera's angle of elevation when the balloon is at an arbitrary height h above the ground. Express θ as a function of h .
 - d. Determine $AV_{[200,275]}$ for θ (as a function of h) and write at least one sentence to carefully explain the meaning of the value you find, including units.
6. Consider a right triangle where the two legs measure 5 and 12 respectively and α is the angle opposite the shorter leg and β is the angle opposite the longer leg.
 - a. What is the exact value of $\cos(\alpha)$?
 - b. What is the exact value of $\sin(\beta)$?
 - c. What is the exact value of $\tan(\beta)$? of $\tan(\alpha)$?
 - d. What is the exact radian measure of α ? approximate measure?
 - e. What is the exact radian measure of β ? approximate measure?
 - f. True or false: for any two angles θ and γ such that $\theta + \gamma = \frac{\pi}{2}$ (radians), it follows that $\cos(\theta) = \sin(\gamma)$.

measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

Using Right Triangles to Evaluate Trigonometric Functions

Figure 1 shows a right triangle with a vertical side of length y and a horizontal side has length x . Notice that the triangle is inscribed in a circle of radius 1. Such a circle, with a center at the origin and a radius of 1, is known as a **unit circle**.

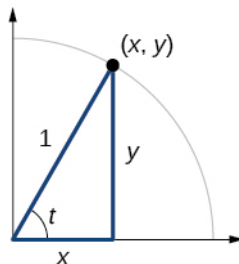


Figure 1

We can define the trigonometric functions in terms an angle t and the lengths of the sides of the triangle. The **adjacent side** is the side closest to the angle, x . (Adjacent means “next to.”) The **opposite side** is the side across from the angle, y . The **hypotenuse** is the side of the triangle opposite the right angle, 1. These sides are labeled in Figure 2.

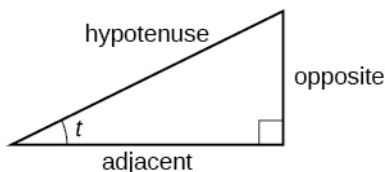


Figure 2 The sides of a right triangle in relation to angle t

Given a right triangle with an acute angle of t , the first three trigonometric functions are listed.

$$\begin{aligned} \text{Sine} \quad \sin t &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \text{Cosine} \quad \cos t &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \text{Tangent} \quad \tan t &= \frac{\text{opposite}}{\text{adjacent}} \end{aligned}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of “**S**ine is **o**pposite over **h**ypotenuse, **C**osine is **a**djacent over **h**ypotenuse, **T**angent is **o**pposite over **a**djacent.”

For the triangle shown in Figure 1, we have the following.

$$\begin{aligned} \sin t &= \frac{y}{1} \\ \cos t &= \frac{x}{1} \\ \tan t &= \frac{y}{x} \end{aligned}$$



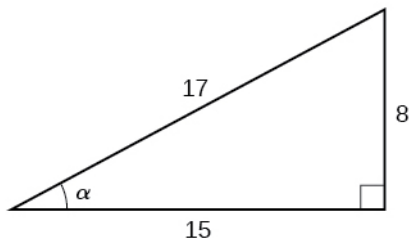
HOW TO

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

EXAMPLE 1**Evaluating a Trigonometric Function of a Right Triangle**

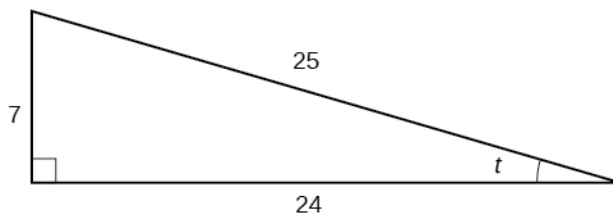
Given the triangle shown in [Figure 3](#), find the value of $\cos \alpha$.

**Figure 3****✓ Solution**

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17.

$$\begin{aligned}\cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17}\end{aligned}$$

> TRY IT #1 Given the triangle shown in [Figure 4](#), find the value of $\sin t$.

**Figure 4****Reciprocal Functions**

In addition to sine, cosine, and tangent, there are three more functions. These too are defined in terms of the sides of the triangle.

$$\begin{array}{ll}\text{Secant} & \sec t = \frac{\text{hypotenuse}}{\text{adjacent}} \\ \text{Cosecant} & \csc t = \frac{\text{hypotenuse}}{\text{opposite}} \\ \text{Cotangent} & \cot t = \frac{\text{adjacent}}{\text{opposite}}\end{array}$$

Take another look at these definitions. These functions are the reciprocals of the first three functions.

$$\begin{array}{ll}\sin t = \frac{1}{\csc t} & \csc t = \frac{1}{\sin t} \\ \cos t = \frac{1}{\sec t} & \sec t = \frac{1}{\cos t} \\ \tan t = \frac{1}{\cot t} & \cot t = \frac{1}{\tan t}\end{array}$$

When working with right triangles, keep in mind that the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in [Figure 5](#). The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

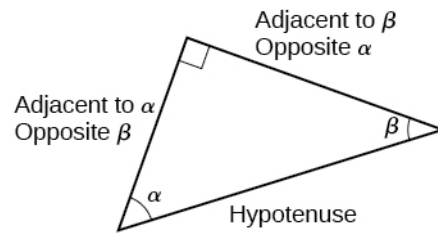


Figure 5 The side adjacent to one angle is opposite the other angle.

Many problems ask for all six trigonometric functions for a given angle in a triangle. A possible strategy to use is to find the sine, cosine, and tangent of the angles first. Then, find the other trigonometric functions easily using the reciprocals.



HOW TO

Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
 - sine as the ratio of the opposite side to the hypotenuse
 - cosine as the ratio of the adjacent side to the hypotenuse
 - tangent as the ratio of the opposite side to the adjacent side
 - secant as the ratio of the hypotenuse to the adjacent side
 - cosecant as the ratio of the hypotenuse to the opposite side
 - cotangent as the ratio of the adjacent side to the opposite side

EXAMPLE 2

Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in [Figure 6](#), evaluate $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, $\csc \alpha$, and $\cot \alpha$.

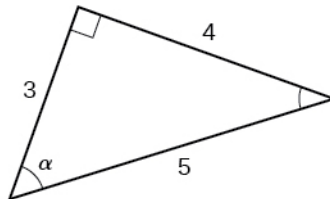


Figure 6

✓ Solution

$$\sin \alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$

$$\tan \alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$

$$\sec \alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$

$$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$

$$\cot \alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

🔍 Analysis

Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.

$$\sec \alpha = \frac{1}{\cos \alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\csc \alpha = \frac{1}{\sin \alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

$$\cot \alpha = \frac{1}{\tan \alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

TRY IT #2 Using the triangle shown in [Figure 7](#), evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

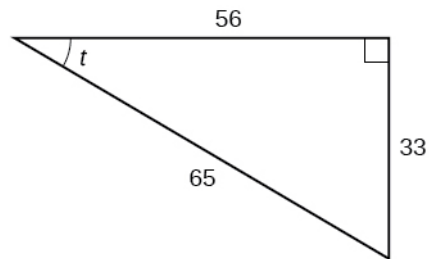


Figure 7

Finding Trigonometric Functions of Special Angles Using Side Lengths

It is helpful to evaluate the trigonometric functions as they relate to the special angles—multiples of 30° , 60° , and 45° . Remember, however, that when dealing with right triangles, we are limited to angles between 0° and 90° .

Suppose we have a 30° , 60° , 90° triangle, which can also be described as a $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ triangle. The sides have lengths in the relation s , $\sqrt{3}s$, $2s$. The sides of a 45° , 45° , 90° triangle, which can also be described as a $\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$ triangle, have lengths in the relation s , s , $\sqrt{2}s$. These relations are shown in [Figure 8](#).

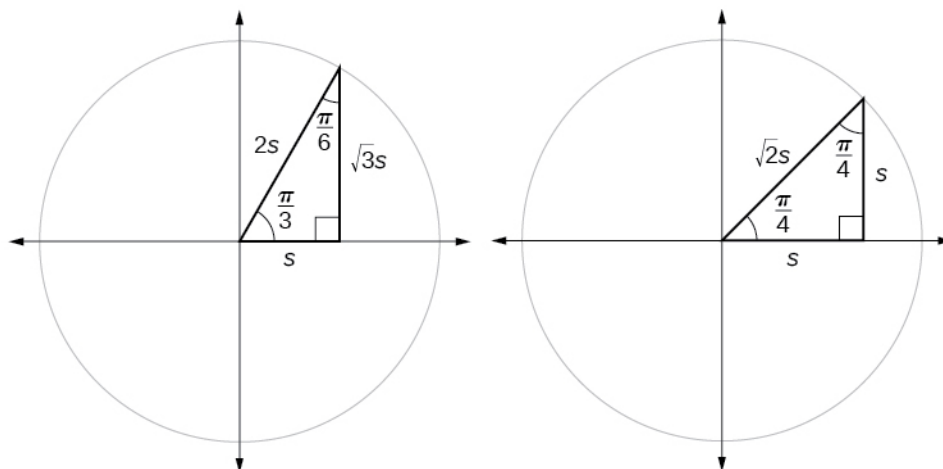


Figure 8 Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.



HOW TO

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in [Figure 8](#) for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

EXAMPLE 3**Evaluating Trigonometric Functions of Special Angles Using Side Lengths**

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

 **Solution**

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$


$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$

$$\csc\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

 **TRY IT #3** Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles, we notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$.

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

See [Figure 9](#).

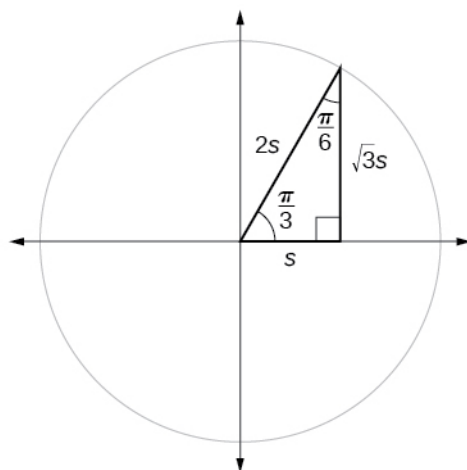
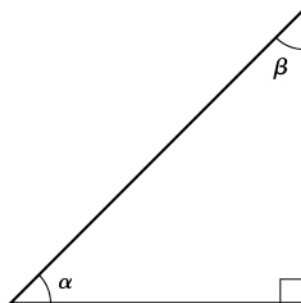


Figure 9 The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from [Figure 9](#), the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other.

Since the three angles of a triangle add to π , and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$ —in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in [Figure 10](#).



$$\begin{aligned}\sin \alpha &= \cos \beta \\ \sin \beta &= \cos \alpha\end{aligned}$$

Figure 10 Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a given angle t , $\cos t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

Cofunction Identities

The **cofunction identities** in radians are listed in [Table 1](#).

$\cos t = \sin\left(\frac{\pi}{2} - t\right)$	$\sin t = \cos\left(\frac{\pi}{2} - t\right)$
$\tan t = \cot\left(\frac{\pi}{2} - t\right)$	$\cot t = \tan\left(\frac{\pi}{2} - t\right)$
$\sec t = \csc\left(\frac{\pi}{2} - t\right)$	$\csc t = \sec\left(\frac{\pi}{2} - t\right)$

Table 1



HOW TO

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

EXAMPLE 4

Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

Solution

According to the cofunction identities for sine and cosine, we have the following.

$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$

So

$$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}$$

> **TRY IT** #4 If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.



HOW TO

Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

EXAMPLE 5

Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in [Figure 11](#).

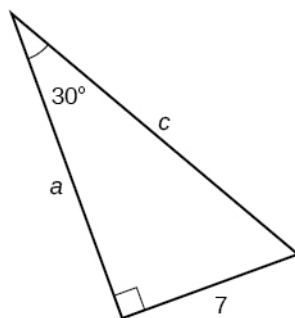


Figure 11

✓ Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a .

$$\begin{aligned} a &= \frac{7}{\tan(30^\circ)} \\ &\approx 12.1 \end{aligned}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c .

$$\begin{aligned}c &= \frac{7}{\sin(30^\circ)} \\ &= 14\end{aligned}$$

> **TRY IT** #5 A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height.

Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See [Figure 12](#).

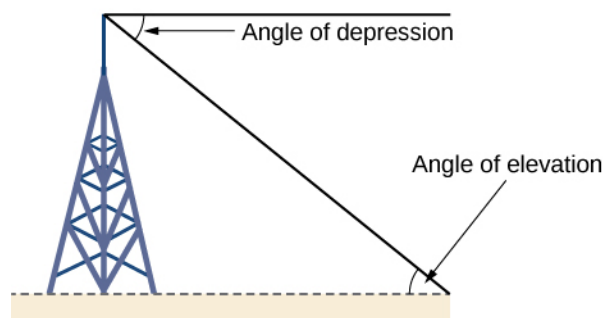


Figure 12



HOW TO

Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

EXAMPLE 6

Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in [Figure 13](#). Find the height of the tree.

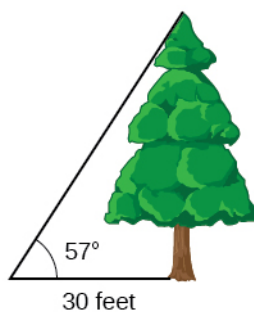


Figure 13

✓ **Solution**

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57° , letting h be the unknown height.

$$\begin{aligned}\tan \theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30 \tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.}\end{aligned}$$

The tree is approximately 46 feet tall.

> **TRY IT** #6 How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

▶ **MEDIA**

Access these online resources for additional instruction and practice with right triangle trigonometry.

[Finding Trig Functions on Calculator \(http://openstax.org/l/findtrigcal\)](http://openstax.org/l/findtrigcal)

[Finding Trig Functions Using a Right Triangle \(http://openstax.org/l/trigttri\)](http://openstax.org/l/trigttri)

[Relate Trig Functions to Sides of a Right Triangle \(http://openstax.org/l/reltrigtri\)](http://openstax.org/l/reltrigtri)

[Determine Six Trig Functions from a Triangle \(http://openstax.org/l/sixtrigfunc\)](http://openstax.org/l/sixtrigfunc)

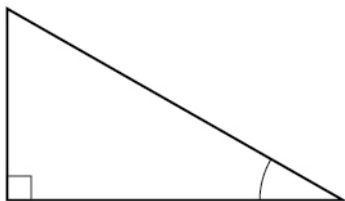
[Determine Length of Right Triangle Side \(http://openstax.org/l/rttrside\)](http://openstax.org/l/rttrside)



7.2 SECTION EXERCISES

Verbal

- For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.
- When a right triangle with a hypotenuse of 1 is placed in a circle of radius 1, which sides of the triangle correspond to the x - and y -coordinates?



3. The tangent of an angle compares which sides of the right triangle?
4. What is the relationship between the two acute angles in a right triangle?
5. Explain the cofunction identity.

Algebraic

For the following exercises, use cofunctions of complementary angles.

6. $\cos(34^\circ) = \sin(\underline{\hspace{1cm}}^\circ)$ 7. $\cos\left(\frac{\pi}{3}\right) = \sin(\underline{\hspace{1cm}})$ 8. $\csc(21^\circ) = \sec(\underline{\hspace{1cm}}^\circ)$
9. $\tan\left(\frac{\pi}{4}\right) = \cot(\underline{\hspace{1cm}})$

For the following exercises, find the lengths of the missing sides if side a is opposite angle A , side b is opposite angle B , and side c is the hypotenuse.

10. $\cos B = \frac{4}{5}, a = 10$ 11. $\sin B = \frac{1}{2}, a = 20$ 12. $\tan A = \frac{5}{12}, b = 6$
13. $\tan A = 100, b = 100$ 14. $\sin B = \frac{1}{\sqrt{3}}, a = 2$ 15. $a = 5, \angle A = 60^\circ$
16. $c = 12, \angle A = 45^\circ$

Graphical

For the following exercises, use [Figure 14](#) to evaluate each trigonometric function of angle A .

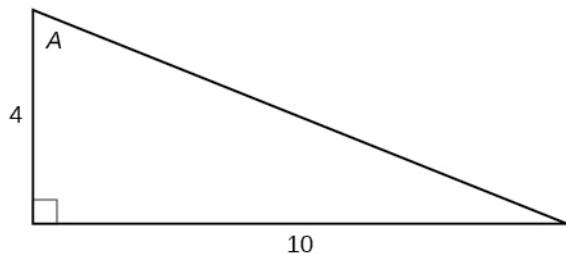


Figure 14

17. $\sin A$ 18. $\cos A$ 19. $\tan A$
20. $\csc A$ 21. $\sec A$ 22. $\cot A$

For the following exercises, use [Figure 15](#) to evaluate each trigonometric function of angle A .

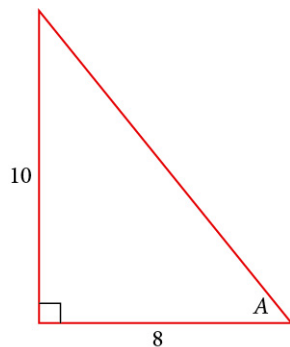
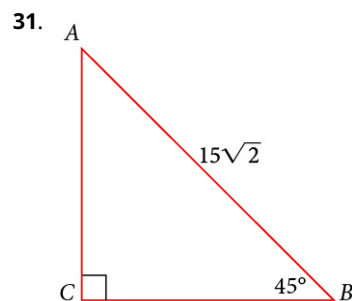
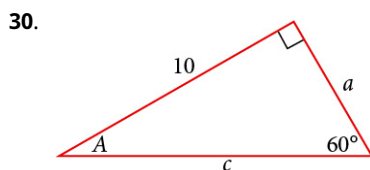
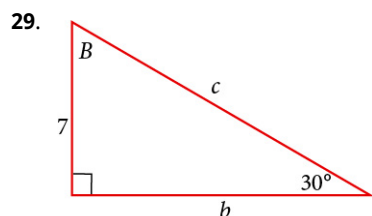


Figure 15

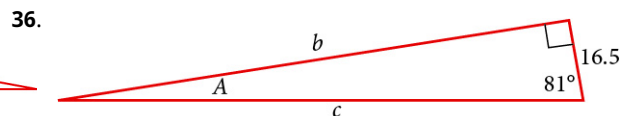
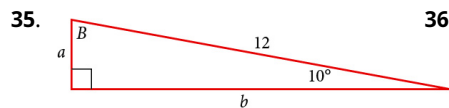
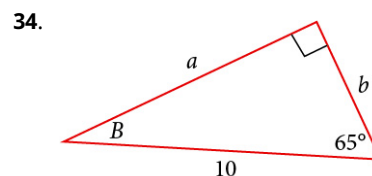
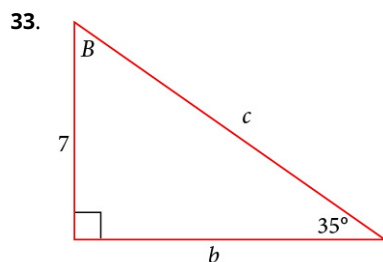
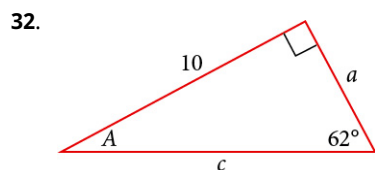
23. $\sin A$ 24. $\cos A$ 25. $\tan A$
 26. $\csc A$ 27. $\sec A$ 28. $\cot A$

For the following exercises, solve for the unknown sides of the given triangle.



Technology

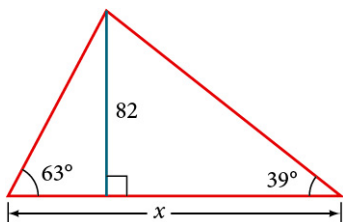
For the following exercises, use a calculator to find the length of each side to four decimal places.



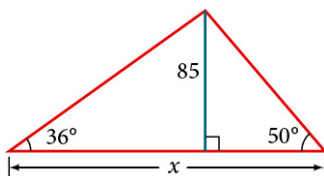
37. $b = 15$, $\sphericalangle B = 15^\circ$ 38. $c = 200$, $\sphericalangle B = 5^\circ$ 39. $c = 50$, $\sphericalangle B = 21^\circ$
 40. $a = 30$, $\sphericalangle A = 27^\circ$ 41. $b = 3.5$, $\sphericalangle A = 78^\circ$

Extensions

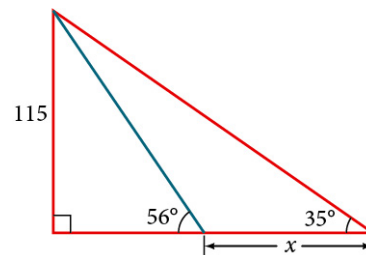
42. Find
- x
- .



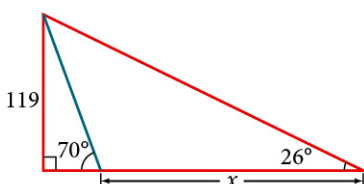
43. Find
- x
- .



44. Find
- x
- .



45. Find
- x
- .



46. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36° , and that the angle of depression to the bottom of the tower is 23° . How tall is the tower?

47. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43° , and that the angle of depression to the bottom of the tower is 31° . How tall is the tower?

48. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15° , and that the angle of depression to the bottom of the monument is 2° . How far is the person from the monument?

49. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18° , and that the angle of depression to the bottom of the monument is 3° . How far is the person from the monument?

50. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40° . From the same location, the angle of elevation to the top of the antenna is measured to be 43° . Find the height of the antenna.

51. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36° . From the same location, the angle of elevation to the top of the lightning rod is measured to be 38° . Find the height of the lightning rod.

Real-World Applications

52. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80° . How high does the ladder reach up the side of the building?
53. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80° . How high does the ladder reach up the side of the building?
54. The angle of elevation to the top of a building in Charlotte is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.
55. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.
56. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60° , how far from the base of the tree am I?

7.3 Unit Circle

Learning Objectives

In this section you will:

- Find function values for the sine and cosine of 30° or $\left(\frac{\pi}{6}\right)$, 45° or $\left(\frac{\pi}{4}\right)$, and 60° or $\left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.



Figure 1 The Singapore Flyer was the world's tallest Ferris wheel until being overtaken by the High Roller in Las Vegas and the Ain Dubai in Dubai. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Ain Dubai, the world's tallest Ferris wheel. Located in Dubai, the most populous city and the financial and tourism hub of the United Arab Emirates, the wheel soars to 820 feet, about 1.5 tenths of a mile. Described as an observation wheel, riders enjoy spectacular views of the Burj Khalifa (the world's tallest building) and the Palm Jumeirah (a human-made archipelago home to over 10,000 people and 20 resorts) as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Trigonometric Functions Using the Unit Circle

We have already defined the trigonometric functions in terms of right triangles. In this section, we will redefine them in

terms of the unit circle. Recall that a unit circle is a circle centered at the origin with radius 1, as shown in [Figure 2](#). The angle (in radians) that t intercepts forms an arc of length s . Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

The x - and y -axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t , we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y) . The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

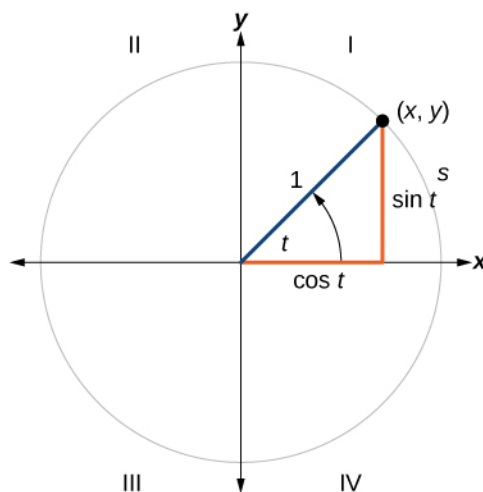


Figure 2 Unit circle where the central angle is t radians

Unit Circle

A **unit circle** has a center at $(0, 0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t .

Let (x, y) be the endpoint on the unit circle of an arc of arc length s . The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

The sine function relates a real number t to the y -coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y -value of the endpoint on the unit circle of an arc of length t . In [Figure 2](#), the sine is equal to y . Like all functions, the **sine function** has an input and an output. Its input is the measure of the angle; its output is the y -coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x -value of the endpoint on the unit circle of an arc of length t . In [Figure 3](#), the cosine is equal to x .

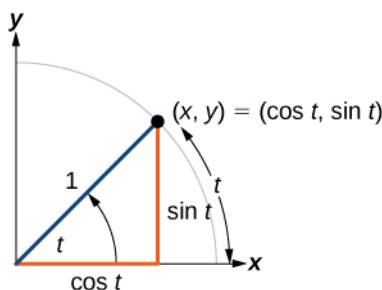


Figure 3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$.

Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

Sine and Cosine Functions

If t is a real number and a point (x, y) on the unit circle corresponds to a central angle t , then

$$\cos t = x$$

$$\sin t = y$$



HOW TO

Given a point $P(x, y)$ on the unit circle corresponding to an angle of t , find the sine and cosine.

1. The sine of t is equal to the y -coordinate of point P : $\sin t = y$.
2. The cosine of t is equal to the x -coordinate of point P : $\cos t = x$.

EXAMPLE 1

Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t , as shown in [Figure 4](#). Find $\cos(t)$ and $\sin(t)$.

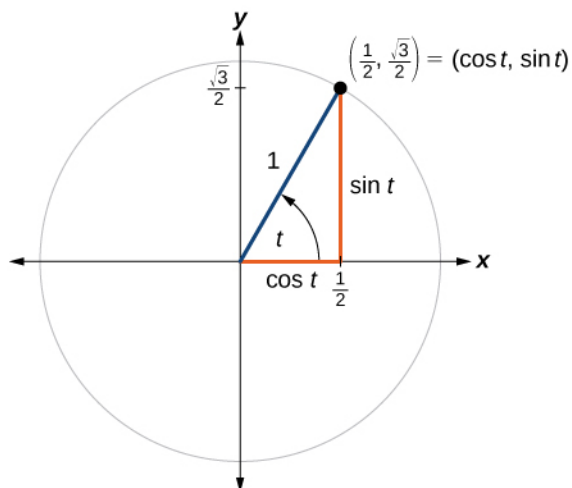


Figure 4

Solution

We know that $\cos t$ is the x -coordinate of the corresponding point on the unit circle and $\sin t$ is the y -coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$

$$y = \sin t = \frac{\sqrt{3}}{2}$$

TRY IT #1 A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in [Figure 5](#). Find $\cos t$ and $\sin t$.

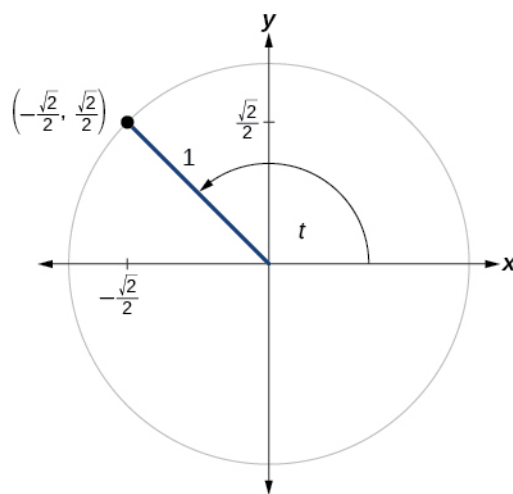


Figure 5

Finding Sines and Cosines of Angles on an Axis

For quadrantal angles, the corresponding point on the unit circle falls on the x - or y -axis. In that case, we can easily calculate cosine and sine from the values of x and y .

EXAMPLE 2

Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

✓ Solution

Moving 90° counterclockwise around the unit circle from the positive x -axis brings us to the top of the circle, where the (x, y) coordinates are $(0, 1)$, as shown in [Figure 6](#).

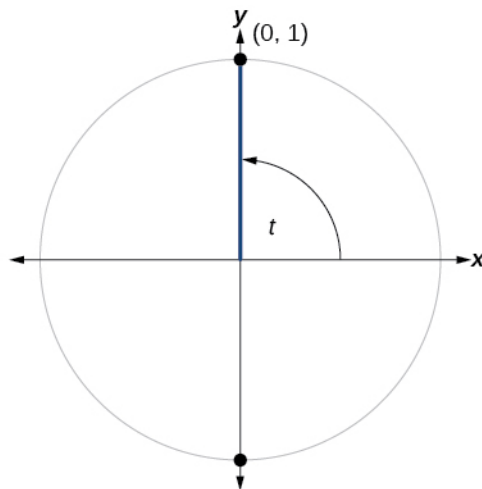


Figure 6

We can then use our definitions of cosine and sine.

$$x = \cos t = \cos(90^\circ) = 0$$

$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

> **TRY IT** #2 Find cosine and sine of the angle π .

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the Pythagorean Identity. See [Figure 7](#).

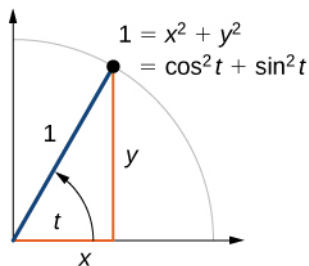


Figure 7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t ,

$$\cos^2 t + \sin^2 t = 1$$



HOW TO

Given the sine of some angle t and its quadrant location, find the cosine of t .

1. Substitute the known value of $\sin t$ into the Pythagorean Identity.
2. Solve for $\cos t$.
3. Choose the solution with the appropriate sign for the x -values in the quadrant where t is located.

EXAMPLE 3

Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

✓ Solution

If we drop a vertical line from the point on the unit circle corresponding to t , we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See [Figure 8](#).

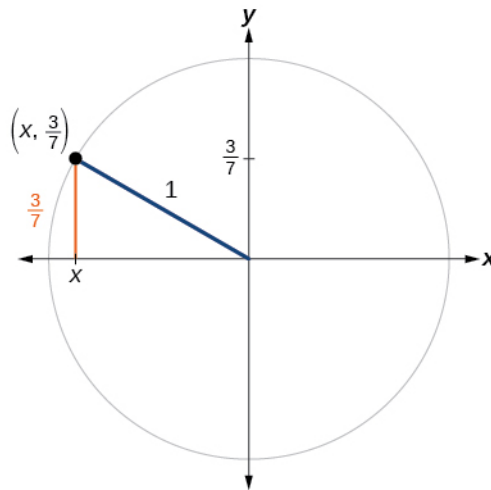


Figure 8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned}\cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}\end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative.

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

> TRY IT #3 If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees, and we found their sines and cosines using right triangles. We can also calculate sines and cosines of the special angles using the Pythagorean Identity.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 9. A 45° – 45° – 90° triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

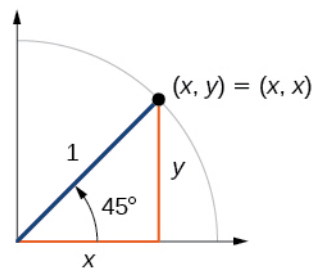


Figure 9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1 so the right triangle formed below the line $y = x$ has sides x

and y ($y = x$), and radius = 1. See [Figure 10](#).

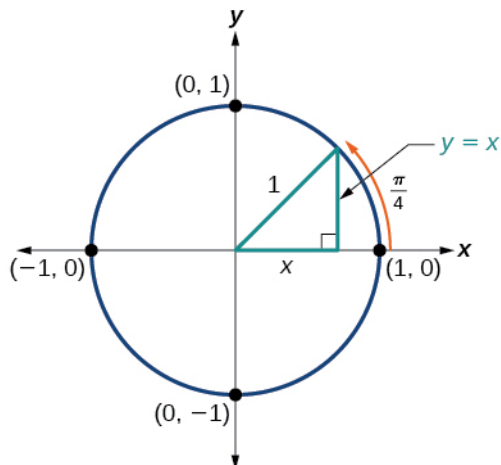


Figure 10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

We can then substitute $y = x$.

$$x^2 + x^2 = 1$$

Next we combine like terms.

$$2x^2 = 1$$

And solving for x , we get

$$\begin{aligned} x^2 &= \frac{1}{2} \\ x &= \pm \frac{1}{\sqrt{2}} \end{aligned}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$\begin{aligned} (x, y) &= (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\ x &= \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} \\ \cos t &= \frac{1}{\sqrt{2}}, \sin t = \frac{1}{\sqrt{2}} \end{aligned}$$

If we then rationalize the denominators, we get

$$\begin{aligned} \cos t &= \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin t &= \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30° , or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at

an angle of 30° , and another at an angle of -30° , as shown in [Figure 11](#). If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60° , as shown in [Figure 12](#).

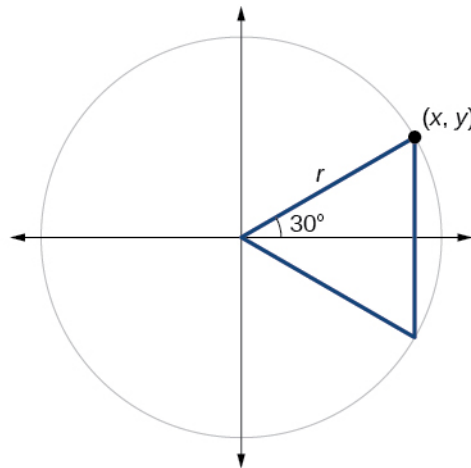


Figure 11

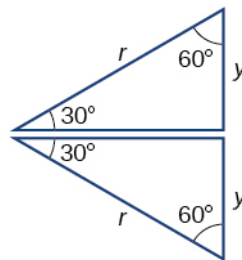


Figure 12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}$$

Use the square root property.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\pm\sqrt{3}}{\pm\sqrt{4}} = \frac{\sqrt{3}}{2}$$

Since y is positive, choose the positive root.

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of

the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD , as shown in [Figure 13](#). Angle A has measure 60° . At point B , we draw an angle ABC with measure of 60° . We know the angles in a triangle sum to 180° , so the measure of angle C is also 60° . Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

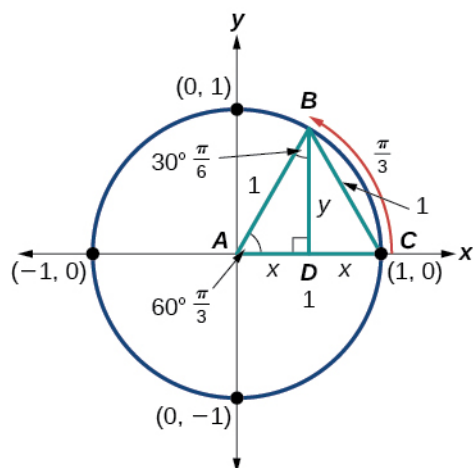


Figure 13

The measure of angle ABD is 30° . Angle ABC is double angle ABD , so its measure is 60° . BD is the perpendicular bisector of AC , so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x -coordinate of point B , which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y , we get

$$\begin{aligned} \frac{1}{4} + y^2 &= 1 \\ y^2 &= 1 - \frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm \frac{\sqrt{3}}{2} \end{aligned}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the y -coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$\begin{aligned} (x, y) &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ x &= \frac{1}{2}, y = \frac{\sqrt{3}}{2} \\ \cos t &= \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2} \end{aligned}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. [Table 1](#) summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
-------	---	---------------------------------	---------------------------------	---------------------------------	---------------------------------

Table 1

Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1

Figure 14 shows the common angles in the first quadrant of the unit circle.

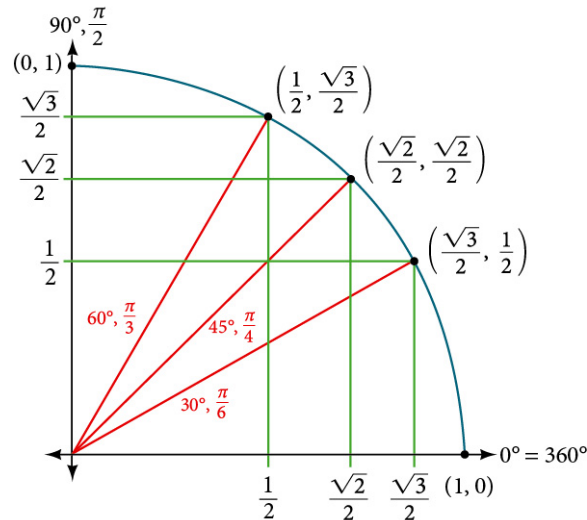


Figure 14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware:** Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.



HOW TO

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key “)”.
4. Press ENTER.

EXAMPLE 4

Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

✓ Solution

Enter the following keystrokes:

$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

© Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sine of 20° , for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

> **TRY IT** #4 Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x , y , and r , there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x -axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in [Figure 15](#). The bounds of the x -coordinate are $[-1, 1]$. The bounds of the y -coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

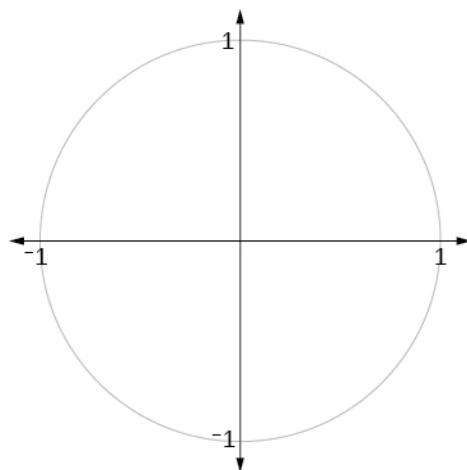


Figure 15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y -coordinate on the unit circle, the other angle with the same sine will share the same y -value, but have the opposite x -value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x -value but will have the opposite y -value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in [Figure 16](#), angle α has the same sine value as angle t ; the cosine values are opposites. Angle β has the same cosine value as angle t ; the sine values are opposites.

$$\begin{array}{ll} \sin(t) = \sin(\alpha) & \text{and} \quad \cos(t) = -\cos(\alpha) \\ \sin(t) = -\sin(\beta) & \text{and} \quad \cos(t) = \cos(\beta) \end{array}$$

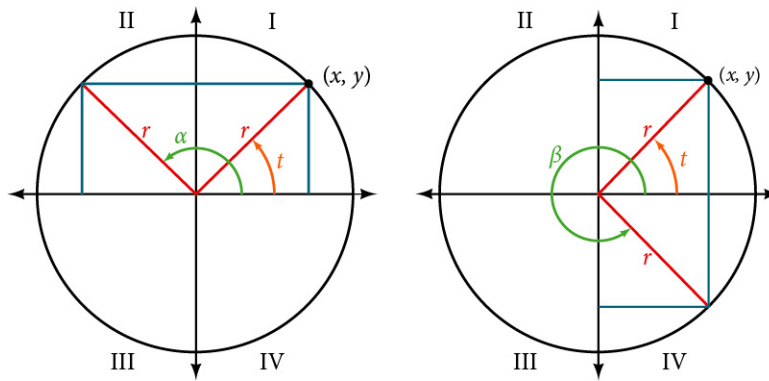


Figure 16

Recall that an angle's reference angle is the acute angle, t , formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90° , or 0 and $\frac{\pi}{2}$ radians. As we can see from [Figure 17](#), for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

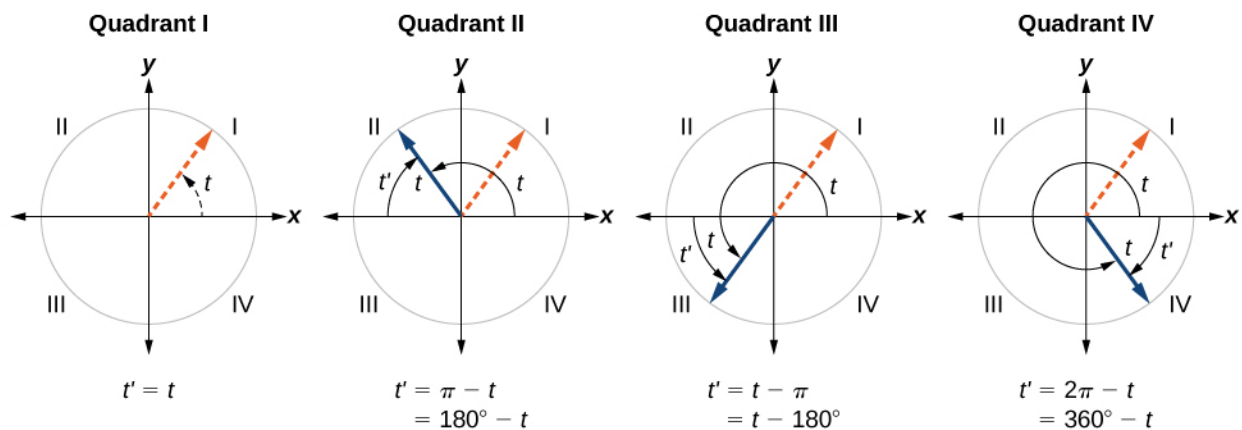


Figure 17

**HOW TO**

Given an angle between 0 and 2π , find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π , add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π .

EXAMPLE 5**Finding a Reference Angle**

Find the reference angle of 225° as shown in [Figure 18](#).

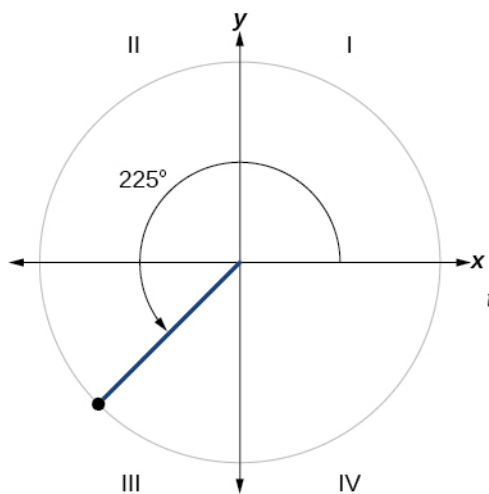


Figure 18

✓ **Solution**

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

> **TRY IT** #5 Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x -values in that quadrant. The sine will be positive or negative depending on the sign of the y -values in that quadrant.

Using Reference Angles to Find Cosine and Sine

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.



HOW TO

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x -values in the quadrant of the original angle.

4. Give the sine the same sign as the y -values in the quadrant of the original angle.

EXAMPLE 6**Using Reference Angles to Find Sine and Cosine**

- (a) Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
 (b) Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

Solution

- (a) 150° is located in the second quadrant. The angle it makes with the x -axis is $180^\circ - 150^\circ = 30^\circ$, so the reference angle is 30° .

This tells us that 150° has the same sine and cosine values as 30° , except for the sign.

$$\cos(30^\circ) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30^\circ) = \frac{1}{2}$$

Since 150° is in the second quadrant, the x -coordinate of the point on the circle is negative, so the cosine value is negative. The y -coordinate is positive, so the sine value is positive.

$$\cos(150^\circ) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150^\circ) = \frac{1}{2}$$

- (b) $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

- TRY IT** #6 (a) Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
 (b) Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in [Figure 19](#). Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

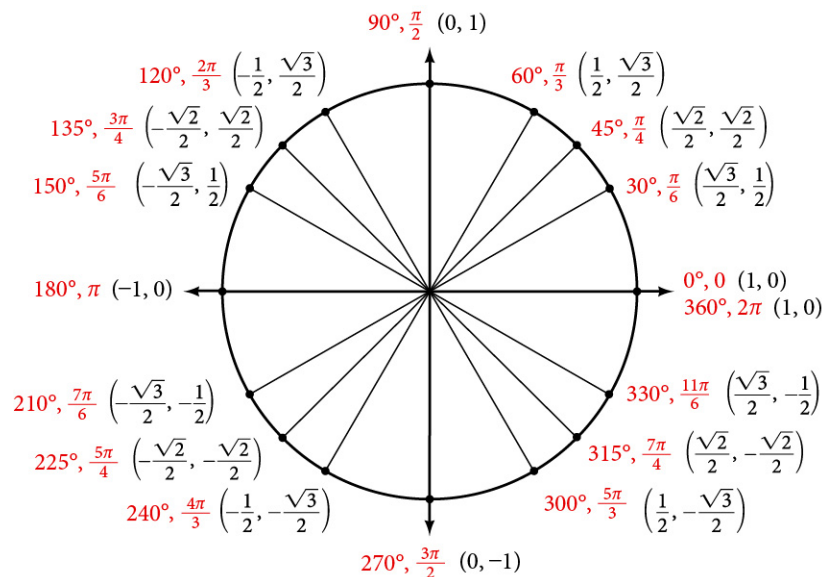


Figure 19 Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$$x = \cos t$$

$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the y - and x -values of the quadrant.



HOW TO

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the x -axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

EXAMPLE 7

Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x -axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π .

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle.


$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\begin{aligned} \cos\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2} \\ \sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2} \end{aligned}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos \theta$ and $y = \sin \theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

 **TRY IT** #7 Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

MEDIA

Access these online resources for additional instruction and practice with sine and cosine functions.

[Trigonometric Functions Using the Unit Circle \(http://openstax.org/l/trigunitcir\)](http://openstax.org/l/trigunitcir)

[Sine and Cosine from the Unit \(http://openstax.org/l/sincosuc\)](http://openstax.org/l/sincosuc)

[Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six \(http://openstax.org/l/sincosmult\)](http://openstax.org/l/sincosmult)

[Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four \(http://openstax.org/l/sincosmult4\)](http://openstax.org/l/sincosmult4)

[Trigonometric Functions Using Reference Angles \(http://openstax.org/l/trigrefang\)](http://openstax.org/l/trigrefang)



7.3 SECTION EXERCISES

Verbal

- Describe the unit circle.
- What do the x- and y-coordinates of the points on the unit circle represent?
- Discuss the difference between a coterminal angle and a reference angle.
- Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.
- Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Algebraic

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

- $\sin(t) < 0$ and $\cos(t) < 0$
- $\sin(t) > 0$ and $\cos(t) > 0$
- $\sin(t) > 0$ and $\cos(t) < 0$
- $\sin(t) > 0$ and $\cos(t) > 0$

For the following exercises, find the exact value of each trigonometric function.

- $\sin \frac{\pi}{2}$
- $\sin \frac{\pi}{3}$
- $\cos \frac{\pi}{2}$
- $\cos \frac{\pi}{3}$
- $\sin \frac{\pi}{4}$
- $\cos \frac{\pi}{4}$
- $\sin \frac{\pi}{6}$
- $\sin \pi$
- $\sin \frac{3\pi}{2}$
- $\cos \pi$
- $\cos 0$
- $\cos \frac{\pi}{6}$
- $\sin 0$

Numeric

For the following exercises, state the reference angle for the given angle.

- 240°
- -170°
- 100°
- -315°
- 135°
- $\frac{5\pi}{4}$
- $\frac{2\pi}{3}$
- $\frac{5\pi}{6}$
- $\frac{-11\pi}{3}$
- $\frac{-7\pi}{4}$
- $\frac{-\pi}{8}$

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

- | | | |
|----------------------|----------------------|----------------------|
| 34. 225° | 35. 300° | 36. 320° |
| 37. 135° | 38. 210° | 39. 120° |
| 40. 250° | 41. 150° | 42. $\frac{5\pi}{4}$ |
| 43. $\frac{7\pi}{6}$ | 44. $\frac{5\pi}{3}$ | 45. $\frac{3\pi}{4}$ |
| 46. $\frac{4\pi}{3}$ | 47. $\frac{2\pi}{3}$ | 48. $\frac{5\pi}{6}$ |
| 49. $\frac{7\pi}{4}$ | | |

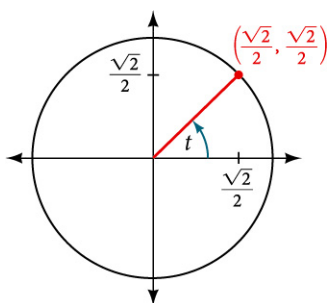
For the following exercises, find the requested value.

- | | | |
|---|--|---|
| 50. If $\cos(t) = \frac{1}{7}$ and t is in the fourth quadrant, find $\sin(t)$. | 51. If $\cos(t) = \frac{2}{9}$ and t is in the first quadrant, find $\sin(t)$. | 52. If $\sin(t) = \frac{3}{8}$ and t is in the second quadrant, find $\cos(t)$. |
| 53. If $\sin(t) = -\frac{1}{4}$ and t is in the third quadrant, find $\cos(t)$. | 54. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of 220° . | 55. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 120° . |
| 56. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$. | 57. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$. | 58. State the domain of the sine and cosine functions. |
| 59. State the range of the sine and cosine functions. | | |

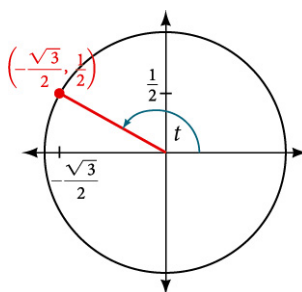
Graphical

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of t .

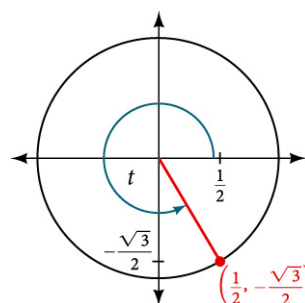
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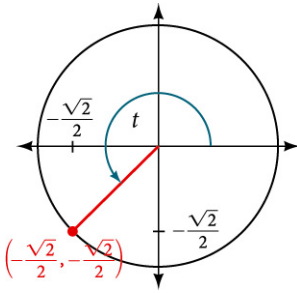
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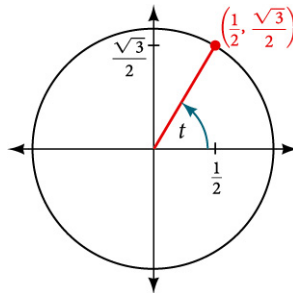
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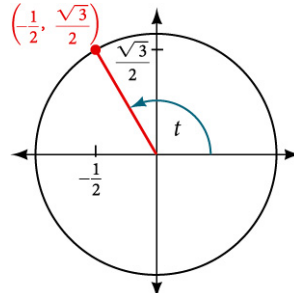
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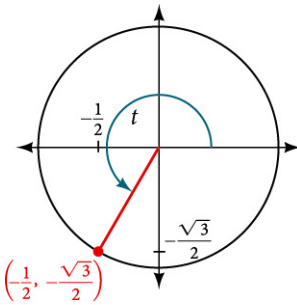
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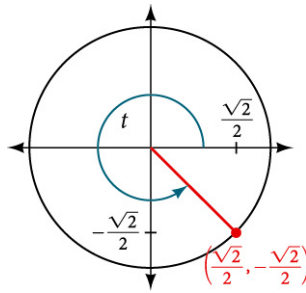
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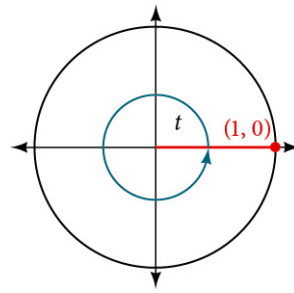
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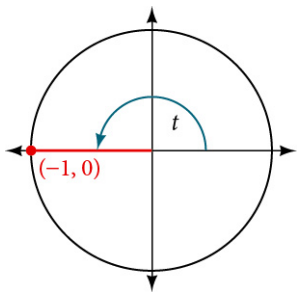
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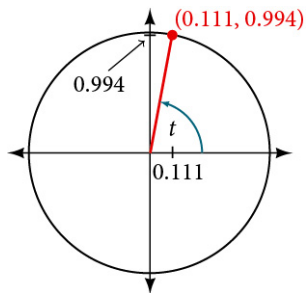
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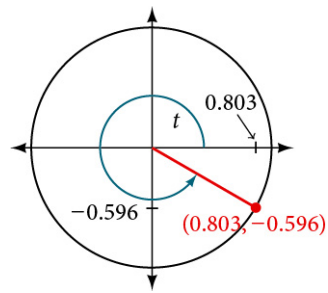
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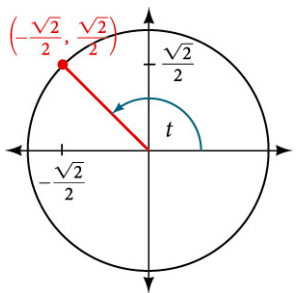
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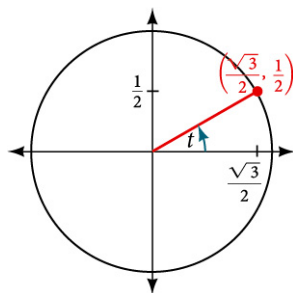
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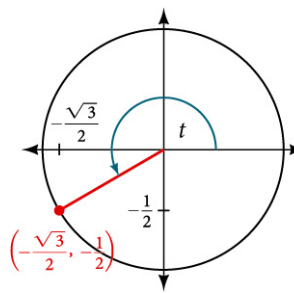
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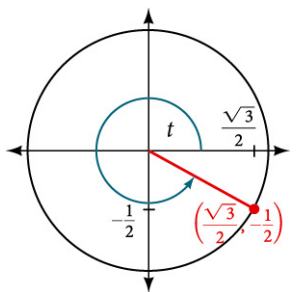
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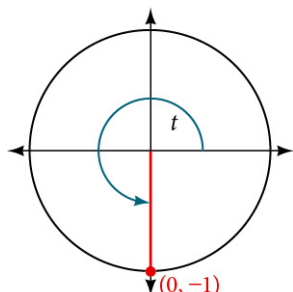
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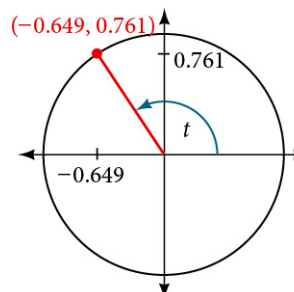
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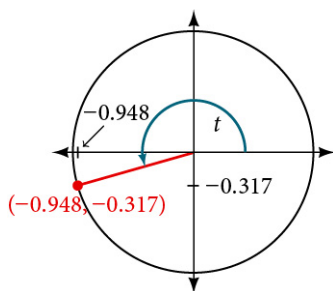
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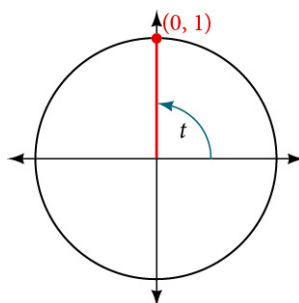
77.



78.



79.



Technology

For the following exercises, use a graphing calculator to evaluate.

80. $\sin \frac{5\pi}{9}$

81. $\cos \frac{5\pi}{9}$

82. $\sin \frac{\pi}{10}$

83. $\cos \frac{\pi}{10}$

84. $\sin \frac{3\pi}{4}$

85. $\cos \frac{3\pi}{4}$

86. $\sin 98^\circ$

87. $\cos 98^\circ$

88. $\cos 310^\circ$

89. $\sin 310^\circ$

Extensions

For the following exercises, evaluate.

90. $\sin \left(\frac{11\pi}{3} \right) \cos \left(\frac{-5\pi}{6} \right)$

91. $\sin \left(\frac{3\pi}{4} \right) \cos \left(\frac{5\pi}{3} \right)$

92. $\sin \left(-\frac{4\pi}{3} \right) \cos \left(\frac{\pi}{2} \right)$

93. $\sin \left(\frac{-9\pi}{4} \right) \cos \left(\frac{-\pi}{6} \right)$

94. $\sin \left(\frac{\pi}{6} \right) \cos \left(\frac{-\pi}{3} \right)$

95. $\sin \left(\frac{7\pi}{4} \right) \cos \left(\frac{-2\pi}{3} \right)$

96. $\cos \left(\frac{5\pi}{6} \right) \cos \left(\frac{2\pi}{3} \right)$

97. $\cos \left(\frac{-\pi}{3} \right) \cos \left(\frac{\pi}{4} \right)$

98. $\sin \left(\frac{-5\pi}{4} \right) \sin \left(\frac{11\pi}{6} \right)$

99. $\sin(\pi) \sin \left(\frac{\pi}{6} \right)$

Real-World Applications

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0, 1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

100. What are the coordinates of the child after 45 seconds?

101. What are the coordinates of the child after 90 seconds?

102. What are the coordinates of the child after 125 seconds?

5.4 graphs of sine and cosine functions (OpenStax Algebra and Trigonometry)

8

PERIODIC FUNCTIONS

Dawn colors the sky over the Olare Motorgi Conservancy bordering the Masai Mara National Reserve in Kenya. (Credit: Modification of "KenyaLive_Day_#02" by Make it Kenya/flickr)

Chapter Outline

- 8.1 Graphs of the Sine and Cosine Functions
- 8.2 Graphs of the Other Trigonometric Functions
- 8.3 Inverse Trigonometric Functions



Introduction to Periodic Functions

The sun has played a core role in many religions. The ancient Egyptian culture portrayed the sun god, Ra (sometimes written as Re), as undertaking a two-part daily journey, with one portion in the sky (day) and the other through the underworld (night). Surya, the Hindu sun god, traces a similar path through the sky on a chariot pulled by seven horses. While their origins and associated narratives are quite different, both Ra and Surya are primary deities and seen as creators and preservers of life. In many Native American cultures, the sun is core to spiritual and religious practice, but is not always a deity. The Sun Dance, practiced differently by many Native American tribes, was a ceremony that generally paid homage to the sun and, in many cases, tested or expressed the strength of the tribe's people.

As one of the most prominent natural phenomena and with its close association to giving life, the sun was an obvious subject for reverence. And its regularity, even in ancient times, made it the primary determinant of time. Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

8.1 Graphs of the Sine and Cosine Functions

Learning Objectives

In this section, you will:

- › Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- › Use phase shifts of sine and cosine curves.



Figure 1 Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on [Trigonometric Functions](http://openstax.org/books/precalculus-2e/pages/5-introduction-to-trigonometric-functions) (<http://openstax.org/books/precalculus-2e/pages/5-introduction-to-trigonometric-functions>), we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x - and y -coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. [Table 1](#) lists some of the values for the sine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Table 1

Plotting the points from the table and continuing along the x -axis gives the shape of the sine function. See [Figure 2](#).

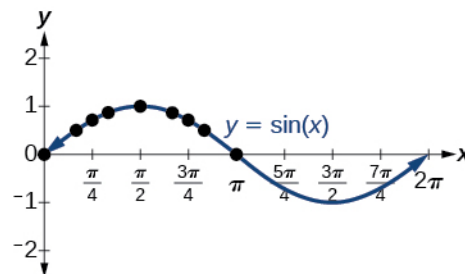


Figure 2 The sine function

Notice how the sine values are positive between 0 and π , which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π , which correspond to the values of the sine function in quadrants III and IV on the unit circle. See [Figure 3](#).

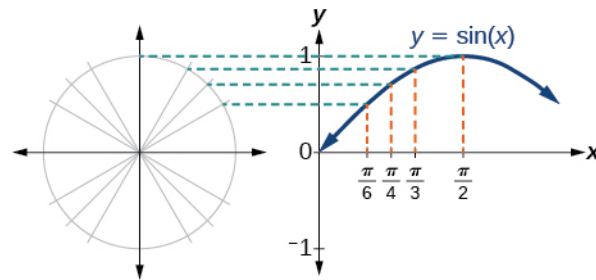


Figure 3 Plotting values of the sine function

Now let's take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. [Table 2](#) lists some of the values for the cosine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 2

As with the sine function, we can plot points to create a graph of the cosine function as in [Figure 4](#).

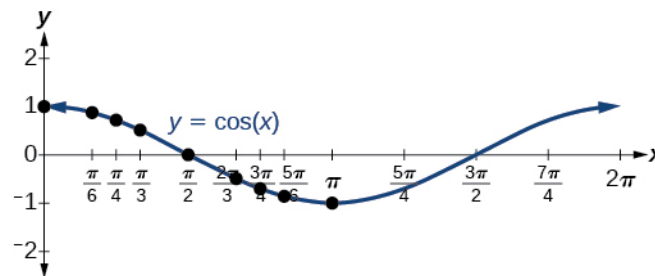


Figure 4 The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π , which means the functions are periodic with a period of 2π . A **periodic function** is a function for which a specific horizontal shift, P , results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f . When this occurs, we call the smallest such horizontal shift with $P > 0$ the period of the function. [Figure 5](#) shows several periods of the sine and cosine functions.

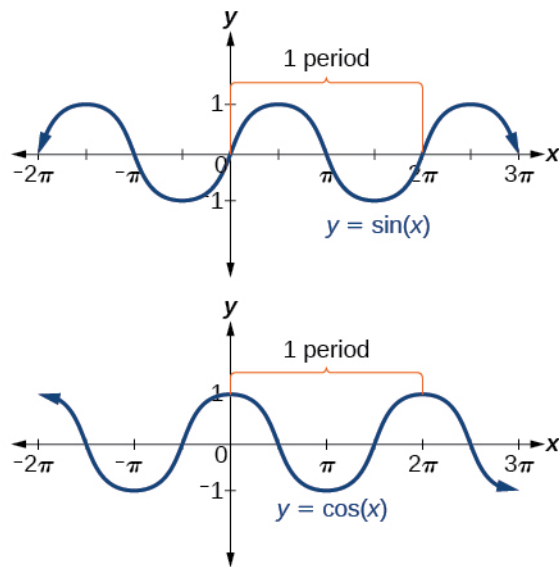


Figure 5

Looking again at the sine and cosine functions on a domain centered at the y -axis helps reveal symmetries. As we can see in [Figure 6](#), the sine function is symmetric about the origin. Recall from [The Other Trigonometric Functions](#) that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin x$. Now we can clearly see this property from the graph.

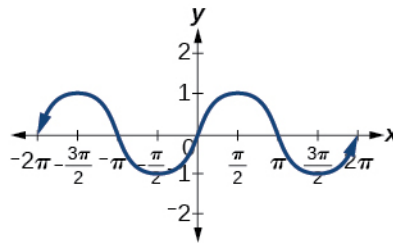


Figure 6 Odd symmetry of the sine function

[Figure 7](#) shows that the cosine function is symmetric about the y -axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos x$.

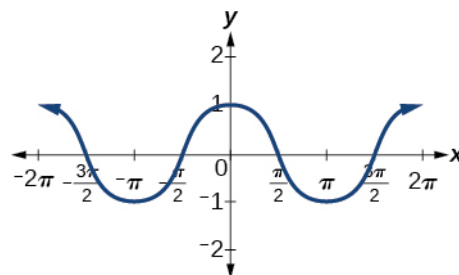


Figure 7 Even symmetry of the cosine function

Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π .
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos x$ is symmetric about the y -axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A \sin(Bx - C) + D$$

and

$$y = A \cos(Bx - C) + D$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π , which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in [Figure 8](#) how the period is indirectly related to $|B|$.

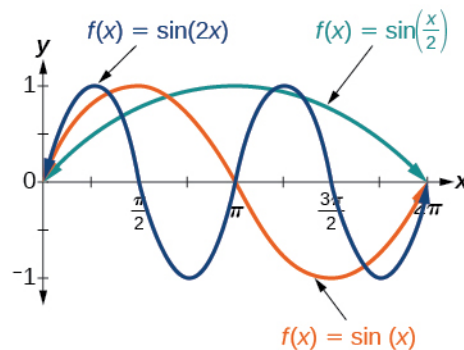


Figure 8

Period of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A \sin(Bx)$$

$$y = A \cos(Bx)$$

The period is $\frac{2\pi}{|B|}$.

EXAMPLE 1

Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

✓ Solution

Let's begin by comparing the equation to the general form $y = A \sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned}
 P &= \frac{2\pi}{|B|} \\
 &= \frac{2\pi}{\frac{\pi}{6}} \\
 &= 2\pi \cdot \frac{6}{\pi} \\
 &= 12
 \end{aligned}$$

> **TRY IT** #1 Determine the period of the function $g(x) = \cos\left(\frac{x}{3}\right)$.

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the horizontal **midline** of the graph, which is the line $y = D$; because $D = 0$ in this case, the midline is the x -axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4 \sin x$ is twice the amplitude of $f(x) = 2 \sin x$. If $|A| < 1$, the function is compressed. [Figure 9](#) compares several sine functions with different amplitudes.

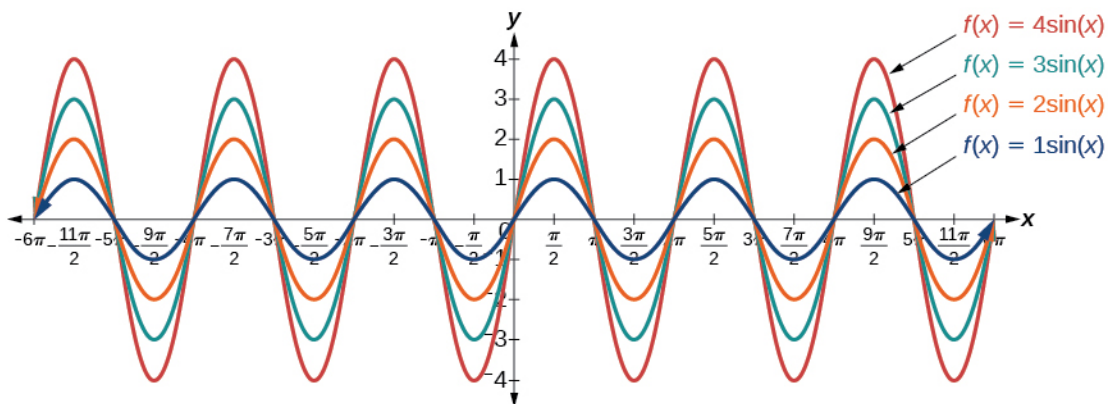


Figure 9

Amplitude of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A \sin(Bx) \text{ and } y = A \cos(Bx)$$

The **amplitude** is $|A|$, which is the vertical height from the **midline**. In addition, notice in the example that

$$|A| = \text{amplitude} = \frac{1}{2} |\text{maximum} - \text{minimum}|$$

EXAMPLE 2

Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4 \sin(x)$? Is the function stretched or compressed vertically?

✓ Solution

Let's begin by comparing the function to the simplified form $y = A \sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

© **Analysis**

The negative value of A results in a reflection across the x -axis of the sine function, as shown in [Figure 10](#).

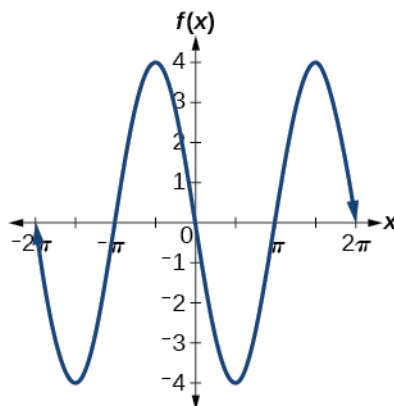


Figure 10

TRY IT #2 What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Analyzing Graphs of Variations of $y = \sin x$ and $y = \cos x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D . Recall the general form:

$$y = A \sin(Bx - C) + D \text{ and } y = A \cos(Bx - C) + D$$

or

$$y = A \sin\left(B\left(x - \frac{C}{B}\right)\right) + D \text{ and } y = A \cos\left(B\left(x - \frac{C}{B}\right)\right) + D$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. [Figure 11](#) shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

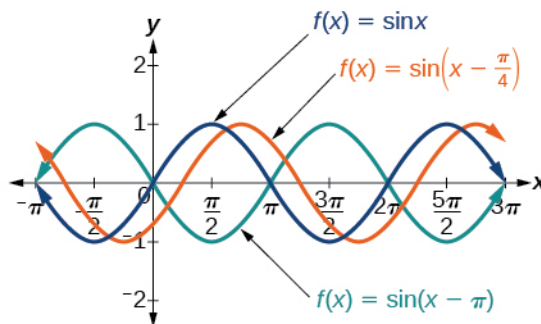


Figure 11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See [Figure 12](#). The function $y = \cos(x) + D$ has its midline at $y = D$.

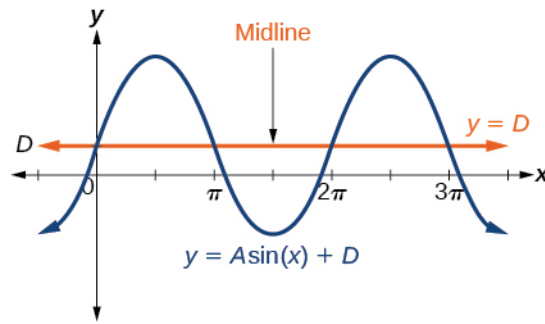


Figure 12

Any value of D other than zero shifts the graph up or down. Figure 13 compares $f(x) = \sin(x)$ with $f(x) = \sin(x) + 2$, which is shifted 2 units up on a graph.

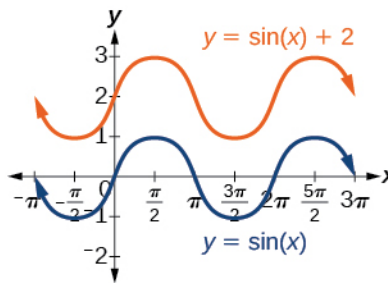


Figure 13

Variations of Sine and Cosine Functions

Given an equation in the form $f(x) = A \sin(Bx - C) + D$ or $f(x) = A \cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the vertical shift.

EXAMPLE 3

Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

☑ Solution

Let's begin by comparing the equation to the general form $y = A \sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

🔍 Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C . Therefore $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

> **TRY IT** #3 Determine the direction and magnitude of the phase shift for $f(x) = 3 \cos\left(x - \frac{\pi}{2}\right)$.

EXAMPLE 4**Identifying the Vertical Shift of a Function**

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

✓ **Solution**

Let's begin by comparing the equation to the general form $y = A \cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

> **TRY IT** #4 Determine the direction and magnitude of the vertical shift for $f(x) = 3 \sin(x) + 2$.

**HOW TO**

Given a sinusoidal function in the form $f(x) = A \sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

EXAMPLE 5**Identifying the Variations of a Sinusoidal Function from an Equation**

Determine the midline, amplitude, period, and phase shift of the function $y = 3 \sin(2x) + 1$.

✓ **Solution**

Let's begin by comparing the equation to the general form $y = A \sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\frac{C}{B} = \frac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

🕒 **Analysis**

Inspecting the graph, we can determine that the period is π , the midline is $y = 1$, and the amplitude is 3. See [Figure 14](#).

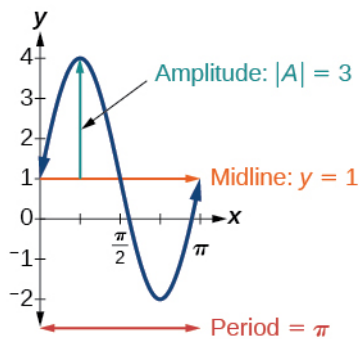


Figure 14

> **TRY IT** #5 Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2} \cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

EXAMPLE 6**Identifying the Equation for a Sinusoidal Function from a Graph**

Determine the formula for the cosine function in [Figure 15](#).

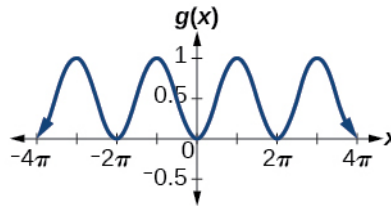


Figure 15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A \sin(Bx - C) + D$$

$$y = A \cos(Bx - C) + D$$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x -axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5 \cos(x) + 0.5$$

TRY IT #6 Determine the formula for the sine function in [Figure 16](#).

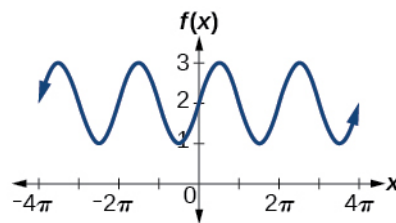


Figure 16

EXAMPLE 7**Identifying the Equation for a Sinusoidal Function from a Graph**

Determine the equation for the sinusoidal function in [Figure 17](#).

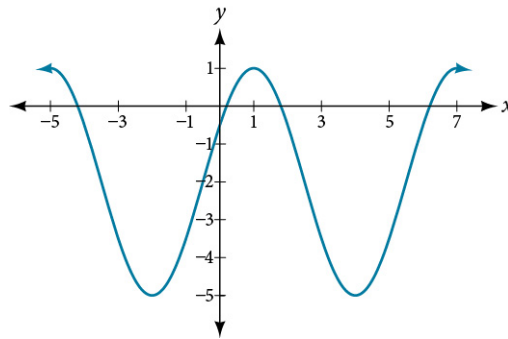


Figure 17

✓ **Solution**

With the highest value at 1 and the lowest value at -5 , the midline will be halfway between at -2 . So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. Therefore, $P = \frac{2\pi}{|B|} = 6$. Using the positive value for B , we find that

$$B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$$

So far, our equation is either $y = 3 \sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3 \cos\left(\frac{\pi}{3}x - C\right) - 2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$y = 3 \cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \text{ or } y = -3 \cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

> **TRY IT** #7 Write a formula for the function graphed in [Figure 18](#).

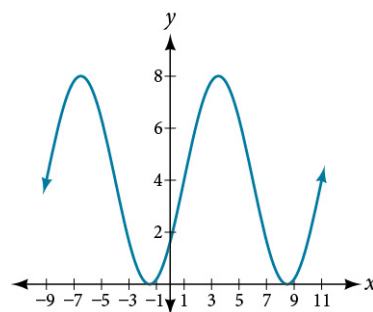


Figure 18

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A \sin(Bx - C) + D \text{ and } y = A \cos(Bx - C) + D,$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.



HOW TO

Given the function $y = A \sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the x -axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the x -axis at $x = \frac{2\pi}{|B|}$.

EXAMPLE 8

Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2 \sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A \sin(Bx)$.

Step 1. We can see from the equation that $A = -2$, so the amplitude is 2.

$$|A| = 2$$

Step 2. The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

Step 3. Because A is negative, the graph descends as we move to the right of the origin.

Step 4–7. The x -intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. [Figure 19](#) shows the graph of the function.

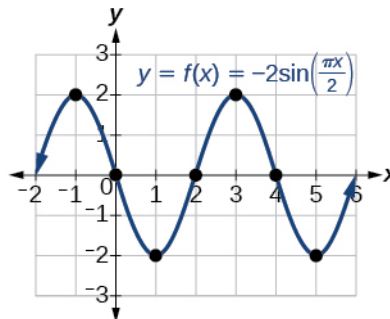


Figure 19

TRY IT #8 Sketch a graph of $g(x) = -0.8 \cos(2x)$. Determine the midline, amplitude, period, and phase shift.



HOW TO

Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form $y = A \sin(Bx - C) + D$ or $y = A \cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $f(x) = A \sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D .

EXAMPLE 9

Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3 \sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

✓ **Solution**

Step 1. The function is already written in general form: $f(x) = 3 \sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$. This graph will have the shape of a sine function, starting at the midline and increasing to the right.

Step 2. $|A| = |3| = 3$. The amplitude is 3.

Step 3. Since $|B| = \left|\frac{\pi}{4}\right| = \frac{\pi}{4}$, we determine the period as follows.

$$P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$$

The period is 8.

Step 4. Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1.$$

The phase shift is 1 unit.

Step 5. [Figure 20](#) shows the graph of the function.

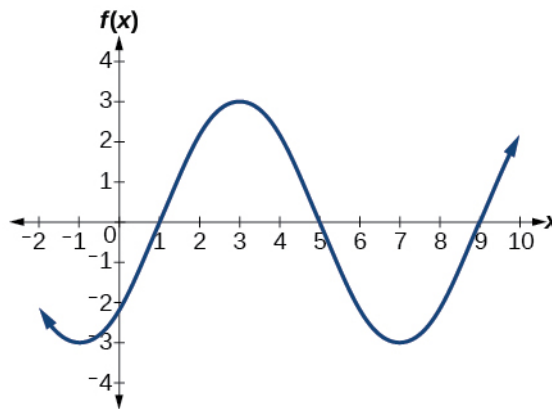


Figure 20 A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

> **TRY IT** #9 Draw a graph of $g(x) = -2 \cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

EXAMPLE 10**Identifying the Properties of a Sinusoidal Function**

Given $y = -2 \cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and vertical shift. Then graph the function.

✓ Solution

Begin by comparing the equation to the general form and use the steps outlined in [Example 9](#).

$$y = A \cos(Bx - C) + D$$

Step 1. The function is already written in general form.

Step 2. Since $A = -2$, the amplitude is $|A| = 2$.

Step 3. $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.

Step 4. $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2 .

Step 5. $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x -axis.

[Figure 21](#) shows one cycle of the graph of the function.

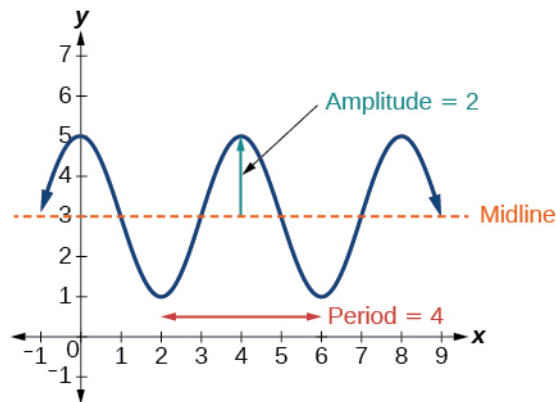


Figure 21

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

EXAMPLE 11**Finding the Vertical Component of Circular Motion**

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y -coordinate of the point as a function of the angle of rotation.

✓ Solution

Recall that, for a point on a circle of radius r , the y -coordinate of the point is $y = r \sin(x)$, so in this case, we get the equation $y(x) = 3 \sin(x)$. The constant 3 causes a vertical stretch of the y -values of the function by a factor of 3, which we can see in the graph in [Figure 22](#).

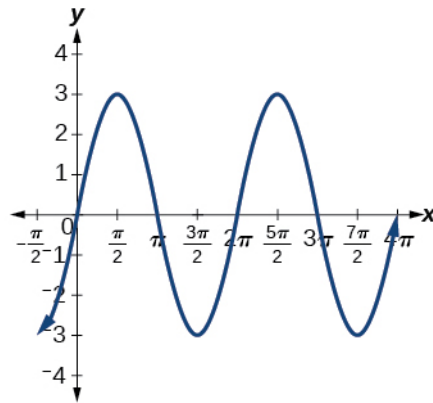


Figure 22

Analysis

Notice that the period of the function is still 2π ; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi, \dots$ Because the outputs of the graph will now oscillate between -3 and 3 , the amplitude of the sine wave is 3 .

TRY IT #10 What is the amplitude of the function $f(x) = 7 \cos(x)$? Sketch a graph of this function.

EXAMPLE 12

Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P , as shown in [Figure 23](#). Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

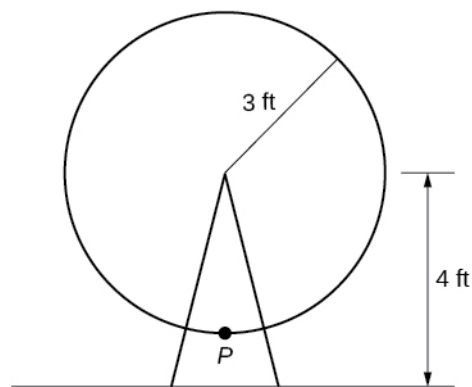


Figure 23

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in [Figure 24](#).

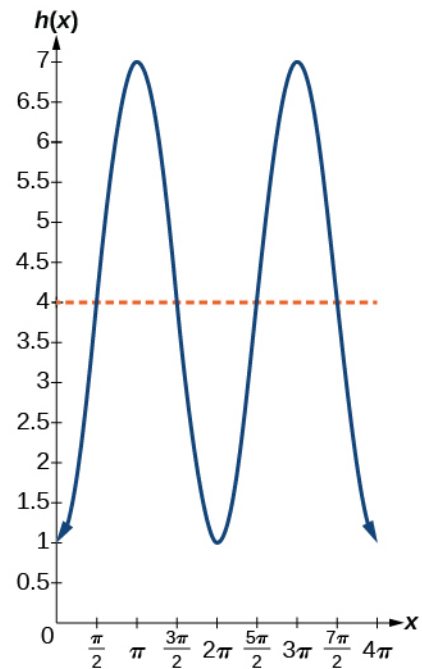


Figure 24

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3 \cos(x) + 4$$

- > **TRY IT** #11 A weight is attached to a spring that is then hung from a board, as shown in [Figure 25](#). As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x . Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x .

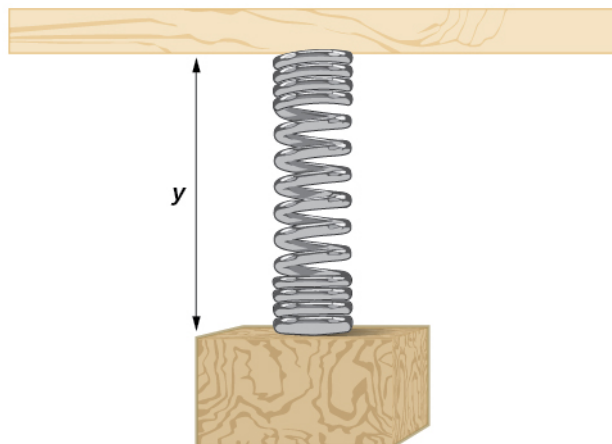


Figure 25

EXAMPLE 13**Determining a Rider's Height on a Ferris Wheel**

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

✓ Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5 \cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

▶ MEDIA

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

[Amplitude and Period of Sine and Cosine \(http://openstax.org/l/ampperiod\)](http://openstax.org/l/ampperiod)

[Translations of Sine and Cosine \(http://openstax.org/l/translasincos\)](http://openstax.org/l/translasincos)

[Graphing Sine and Cosine Transformations \(http://openstax.org/l/transformsincos\)](http://openstax.org/l/transformsincos)

[Graphing the Sine Function \(http://openstax.org/l/graphsinefunc\)](http://openstax.org/l/graphsinefunc)

**8.1 SECTION EXERCISES****Verbal**

1. Why are the sine and cosine functions called periodic functions?
2. How does the graph of $y = \sin x$ compare with the graph of $y = \cos x$? Explain how you could horizontally translate the graph of $y = \sin x$ to obtain $y = \cos x$.
3. For the equation $A \cos(Bx + C) + D$, what constants affect the range of the function and how do they affect the range?
4. How does the range of a translated sine function relate to the equation $y = A \sin(Bx + C) + D$?
5. How can the unit circle be used to construct the graph of $f(t) = \sin t$?

Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y -values and their corresponding x -values on one period for $x > 0$. Round answers to two decimal places if necessary.

6. $f(x) = 2 \sin x$

7. $f(x) = \frac{2}{3} \cos x$

8. $f(x) = -3 \sin x$

9. $f(x) = 4 \sin x$

10. $f(x) = 2 \cos x$

11. $f(x) = \cos(2x)$

12. $f(x) = 2 \sin\left(\frac{1}{2}x\right)$

13. $f(x) = 4 \cos(\pi x)$

14. $f(x) = 3 \cos\left(\frac{6}{5}x\right)$

15. $y = 3 \sin(8(x + 4)) + 5$

16. $y = 2 \sin(3x - 21) + 4$

17. $y = 5 \sin(5x + 20) - 2$

For the following exercises, graph one full period of each function, starting at $x = 0$. For each function, state the amplitude, period, and midline. State the maximum and minimum y -values and their corresponding x -values on one period for $x > 0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. $f(t) = 2 \sin\left(t - \frac{5\pi}{6}\right)$

19. $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

20. $f(t) = 4 \cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21. $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

22. $f(x) = 4 \sin\left(\frac{\pi}{2}(x - 3)\right) + 7$

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in [Figure 26](#).

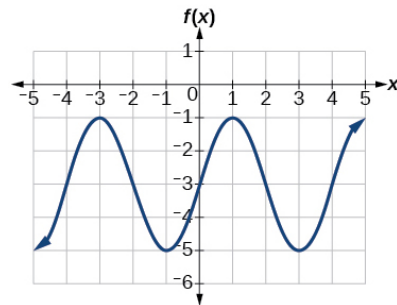


Figure 26

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [Figure 27](#).

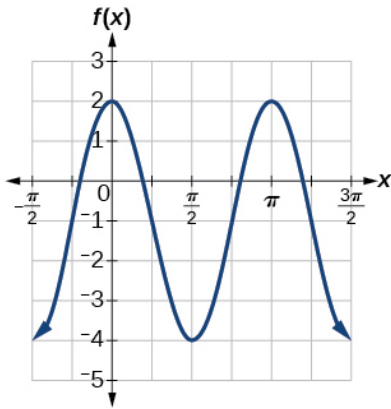


Figure 27

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [Figure 28](#).

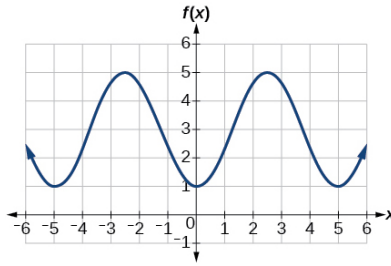


Figure 28

26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [Figure 29](#).

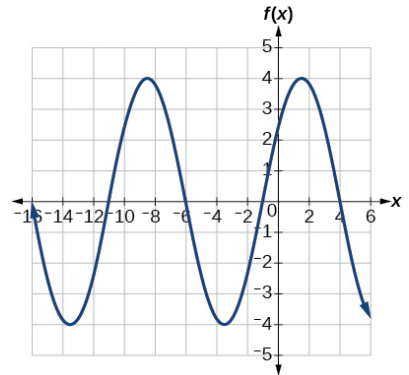


Figure 29

27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [Figure 30](#).

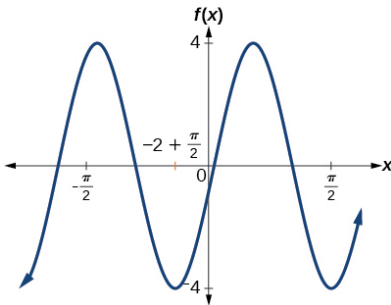


Figure 30

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [Figure 31](#).

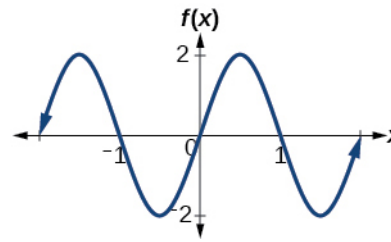


Figure 31

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [Figure 32](#).

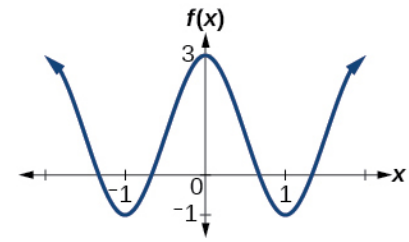


Figure 32

30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [Figure 33](#).

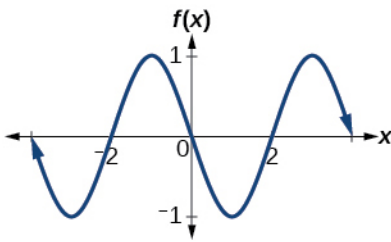


Figure 33

Algebraic

For the following exercises, let $f(x) = \sin x$.

31. On $[0, 2\pi)$, solve $f(x) = 0$. 32. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.
33. Evaluate $f\left(\frac{\pi}{2}\right)$.
34. On $[0, 2\pi)$, $f(x) = \frac{\sqrt{2}}{2}$. Find all values of x . 35. On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x -value(s)?
36. On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x -value(s)?
37. Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to _____.

For the following exercises, let $f(x) = \cos x$.

38. On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$. 39. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.
40. On $[0, 2\pi)$, find the x -intercepts of $f(x) = \cos x$.
41. On $[0, 2\pi)$, find the x -values at which the function has a maximum or minimum value. 42. On $[0, 2\pi)$, solve the equation $f(x) = \frac{\sqrt{3}}{2}$.

Technology

43. Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does. 44. Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?
45. Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = \sin x$.
46. Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows. 47. Graph $f(x) = \frac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$ and explain what the graph shows.

6 Topics for Integration

6.1 Sigma notation (Active Calculus)

6.2 additional practice with geometry of definite integrals (TBIL)

Calculus Fun Fact: Calculus played a key role in the development of navigation in the 17th and 18th centuries

6.1 Sigma notation (Active Calculus)

4.2.1 Sigma Notation

We have used sums of areas of rectangles to approximate the area under a curve. Intuitively, we expect that using a larger number of thinner rectangles will provide a better estimate for the area. Consequently, we anticipate dealing with sums of a large number of terms. To do so, we introduce *sigma notation*, named for the Greek letter Σ , which is the capital letter S in the Greek alphabet.

For example, say we are interested in the sum

$$1 + 2 + 3 + \cdots + 100,$$

the sum of the first 100 natural numbers. In sigma notation we write

$$\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 100.$$

We read the symbol $\sum_{k=1}^{100} k$ as “the sum from k equals 1 to 100 of k .” The variable k is called the index of summation, and any letter can be used for this variable. The pattern in the terms of the sum is denoted by a function of the index; for example,

$$\sum_{k=1}^{10} (k^2 + 2k) = (1^2 + 2 \cdot 1) + (2^2 + 2 \cdot 2) + (3^2 + 2 \cdot 3) + \cdots + (10^2 + 2 \cdot 10),$$

and more generally,

$$\sum_{k=1}^n f(k) = f(1) + f(2) + \cdots + f(n).$$

Sigma notation allows us to vary easily the function being used to describe the terms in the sum, and to adjust the number of terms in the sum simply by changing the value of n . We test our understanding of this new notation in the following activity.

Activity 4.2.2. For each sum written in sigma notation, write the sum long-hand and evaluate the sum to find its value. For each sum written in expanded form, write the sum in sigma notation.

a. $\sum_{k=1}^5 (k^2 + 2)$

d. $4 + 8 + 16 + 32 + \cdots + 256$

b. $\sum_{i=3}^6 (2i - 1)$

c. $3 + 7 + 11 + 15 + \cdots + 27$

e. $\sum_{i=1}^6 \frac{1}{2^i}$

4.2.2 Riemann Sums

When a moving body has a positive velocity function $y = v(t)$ on a given interval $[a, b]$, the area under the curve over the interval gives the total distance the body travels on $[a, b]$. We are also interested in finding the exact area bounded by $y = f(x)$ on an interval $[a, b]$,

6.2 additional practice with geometry of definite integrals (TBIL)

- d Find the intersection of two graphs. ([Khan Academy](#)⁵)
- e Find the area of plane shapes, such as rectangles, triangles, circles, and trapezoids. ([Math is fun](#)⁶)

4.1 Geometry of definite integrals (IN1)

Learning Outcomes

- Use geometric formulas to compute definite integrals.

4.1.1 Activities

Definition 4.1.1 The **definite integral** for a positive function $f(x) \geq 0$ between the points $x = a$ and $x = b$ is the area between the function and the x -axis. We denote this quantity as $\int_a^b f(x) dx$ \diamond

Remark 4.1.2 For some functions which have known geometric shapes (like pieces of lines or circles) we can already compute these area exactly and we will do so in this section. But for most functions we do not know quite yet how to compute these areas. In the next section, we will see that because we can compute the areas of rectangles quite easily, we can always try to approximate a shape with rectangles, even if this could be a very coarse approximation.

Activity 4.1.3 Consider the linear function $f(x) = 2x$. Sketch a graph of this function. Consider the area between the x -axis and the function on the interval $[0, 1]$. What is $\int_0^1 f(x) dx$?

- A. 1 C. 3
B. 2 D. 4

Activity 4.1.4 Consider the linear function $f(x) = 4x$. What is $\int_0^1 f(x) dx$?

- A. 1 C. 3
B. 2 D. 4

Activity 4.1.5 Consider the linear function $f(x) = 2x + 2$. Notice that on the interval $[0, 1]$, the shape formed between the graph and the x -axis is a trapezoid. What is $\int_0^1 f(x) dx$?

- A. 1 C. 3
B. 2 D. 4

Activity 4.1.6 Consider the function $f(x) = \sqrt{4 - x^2}$. Notice that on the domain $[-2, 2]$, the shape formed between the graph and the x -axis is a semicircle. What is $\int_{-2}^2 f(x) dx$?

- A. π C. 3π
B. 2π D. 4π

Definition 4.1.7 If a function $f(x) \leq 0$ on $[a, b]$, then we define the integral

⁵www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:eq/x2ec2f6f830c9fb89:quad-sys/v/line-and-parabola-system

⁶www.mathsisfun.com/area.html

between a and b to be

$$\int_a^b f(x) dx = (-1) \times \text{area between the graph of } f \text{ and the } x\text{-axis on the interval } [a, b].$$

So the definite integral for a negative function is the "negative" of the area between the graph and the x -axis. \diamond

Activity 4.1.8 Explain how to use geometric formulas for area to compute the following definite integrals. For each part, sketch the function to support your explanation.

1.

$$\int_1^6 (-3x + 6) dx$$

2.

$$\int_2^6 (-3x + 6) dx$$

3.

$$\int_1^5 \left(-\sqrt{-(x-1)^2 + 16} \right) dx$$

Activity 4.1.9 The graph of $g(t)$ and the areas A_1, A_2, A_3 are given below.

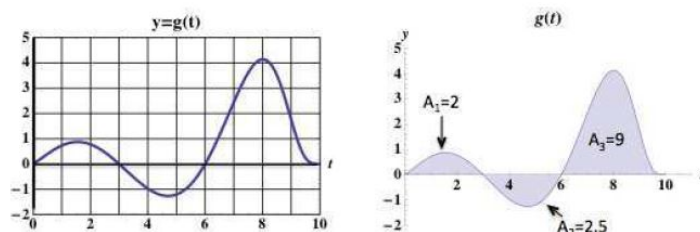


Figure 80

(a) Find $\int_3^3 g(t) dt$

(b) Find $\int_3^6 g(t) dt$

(c) Find $\int_0^{10} g(t) dt$

- (d) Suppose that $g(t)$ gives the velocity in fps at time t (in seconds) of a particle moving in the vertical direction. A positive velocity indicates that the particle is moving up, a negative velocity indicates that the particle is moving down. If the particle started at a height of 3ft, at what height would it be after 3 seconds? After 6 seconds? After 10 seconds? At what time does the particle reach the highest point in this time interval?

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