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Klein-Four Covers of the Projective Line in Characteristic Two

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1 Introduction

There are many ways to stratify the moduli space of curves. In characteristic $p > 0$, one of the most natural stratifications comes from looking at the p-ranks of the curves. The p-rank of a curve X (or, more precisely, the p-rank of its Jacobian) can be defined as $\dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, \mathrm{Jac}(X))$ where μ_p is the kernel of Frobenius on $\mathbb{G}_m$. In particular, curves of p-rank σ will have precisely p^σ distinct p-torsion points on their Jacobian.

There is an idea that curves that have lots of automorphisms should have small p-rank – in particular, the automorphisms would have to permute the p-torsion points and this would lead to a restriction on the possible number of these points. This idea has never been precisely put into the form of a conjecture or theorem, but several attempts have been made to investigate the relationship between automorphism groups and p-ranks.

It follows from [2] in characteristic $p > 2$ and [9] in characteristic 2 that hyperelliptic curves behave similar to generic curves in the sense that there exist curves of each possible 2-rank. In this note, we investigate what one can say about the 2-ranks of curves which have multiple copies of $\mathbb{Z}/2\mathbb{Z}$ in their automorphism group. More precisely, we consider curves defined over an algebraically closed field of characteristic $p = 2$ which admit an action of $(\mathbb{Z}/2\mathbb{Z})^2$ and such that their quotient by this action is $\mathbb{P}^1$.

In Section Two of this paper, we introduce notation and recall some results from [2] and [3] about the moduli space of Klein-four covers of the projective line. We also recall some results from the theory of Artin-Schreier covers that will be used to compute the genera and 2-ranks of the relevant curves. Section Three is concerned with some nonexistence results, and we prove a series of results about when various 2-ranks do not occur.

In the fourth section, we prove that section three covered all possible obstructions, and in particular we prove (a stronger version of) the following theorem.

Theorem 1.1 *Let $g \geq 0$ and $0 \leq \sigma \leq g$. Then there exists a curve X with $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \subseteq \mathrm{Aut}(X)$ and $X/G \cong \mathbb{P}^1$ such that X has genus g and 2-rank σ unless $\sigma = g-1$ or g is even and $\sigma = 1$.*

It will follow from the constructions of these curves that they are all defined over the finite field $\mathbb{F}_4$ and in most cases they can be chosen to be defined over $\mathbb{F}_2$.

We also relate our results to a result of Zhu in [9]. In particular, she proves that there exist hyperelliptic curves of every possible 2-rank with no extra automorphisms, while the following theorem shows precisely when a hyperelliptic curve can have extra involutions.

Theorem 1.2 *There are hyperelliptic curves of genus g and 2-rank σ which contain an additional involution in their automorphism group if and only if $g \equiv \sigma \pmod 2$.*

2 Notation and Machinery

In this article, we work over an algebraically closed field k of characteristic $p = 2$. We wish to examine curves that are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of the projective line $\mathbb{P}^1_k$. In [2], we examined such curves defined over algebraically closed fields of characteristic $p > 2$ and in particular we used such curves to construct hyperelliptic curves with particular group schemes arising as the p-torsion of their Jacobians. When the characteristic of k is not equal to two, this Hurwitz space of such covers is well-defined (for details, see the results of Wewer in [8]) and in [2] we denoted the moduli space of genus g curves which are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\mathbb{P}^1$ by $\mathcal{H}_{g,2}$. However, when the characteristic of k is equal to two we are in the situation of wild ramification, and Wewer's results do not hold. In particular, it is not clear whether $\mathcal{H}_{g,2}$ will be well-defined as a smooth moduli space (see [5] and [6] for details on some of the aspects that can go wrong when defining Hurwitz Spaces associated to wild ramification). In this note we will abuse notation and define $\mathcal{H}_{g,2}$ merely as the set of all $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\mathbb{P}^1$ when the characteristic of k equals two.

The key result which we will make use of in this paper is the following theorem which follows immediately from a result of Kani and Rosen [4].

Theorem 2.1 *Let X be a curve in $\mathcal{H}_{g,2}$ and let H_1, H_2, and $H_{1,2}$ be the three subgroups of the group $(\mathbb{Z}/2\mathbb{Z})^2$ with respect to a fixed basis. Furthermore, let C_1, C_2, and $C_{1,2}$ be the three quotient curves of X by these subgroups. Then*

$$\mathrm{Jac}(X) \sim \prod \mathrm{Jac}(C_S)$$

In particular, if g_S is the genus of C_S and σ_S is the p-rank of C_S then we have that $g_X = g_1 + g_2 + g_{1,2}$ and $\sigma_X = \sigma_1 + \sigma_2 + \sigma_{1,2}$.

In the case where the characteristic of k is not equal to 2, we were able to show that one could further deduce from the fact that the degree of this isogeny is a power of two some results about invariants such as the a-number. While we cannot do this in the case under consideration in this paper, there is more that we can say. In particular, we know that C_1 and C_2 must be Artin-Schreier covers, and therefore can be put into the form $C_i : y^2 + y = f_i(x)$ where f_i is a rational function all of whose poles are of odd order. In this case, it follows from results of van der Geer and van der Vlugt in [7] that the third quotient is of the form $C_{1,2} : y^2 + y = f_1(x) + f_2(x)$.

The following results about the genus and p-rank of Artin-Schreier curves in characteristic two are well known and follow from the Riemann-Hurwitz and Deuring-Shafarevich formulae, and will be used throughout this note without reference.

Theorem 2.2 *Let $y^2 + y = f(x)$ define a hyperelliptic curve C in characteristic two. Let $f(x)$ have k poles given by $x_1, \ldots, x_k$ and let n_i be the order of the pole at x_i. Without loss of generality we can assume that all of the n_i are odd. Then the genus of C is given by the formula $-1 + \frac{1}{2}\sum(n_i + 1)$ and the 2-rank of C is given by $k - 1$.*

To conclude this introduction we define the type of a curve $X \in \mathcal{H}_{g,2}$ to be the unordered triple $\{g_1, g_2, g_3\}$ consisting of the genera of the three $\mathbb{Z}/2\mathbb{Z}$ quotients of X. In particular, it follows that the g_i are integers such that $0 \leq g_i \leq \frac{g+1}{2}$ and $g_1 + g_2 + g_3 = g$. In [3] we show that, when the characteristic of k is *not* equal to two, the irreducible components of $\mathcal{H}_{g,2}$ correspond to the set of curves of a given type. However, as discussed above, when the characteristic of k is equal to two, the objects $\mathcal{H}_{g,2}$ may not be well-defined as geometric objects. For a given partition $\mathfrak{p}$ of g satisfying the necessary conditions we will again abuse notation and define $\mathcal{H}_{g,2,\mathfrak{p}}$ to be the set of all Klein-four covers of $\mathbb{P}^1$ of type $\mathfrak{p}$.

We note that the type of a curve is technically the type of the cover $X \to \mathbb{P}^1$, and in a small number of cases a curve X can be considered a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{P}^1$ in more than one way leading to different types. We show in [3] that this is rare in characteristic $p \neq 2$ (and happens exactly in the case where $1 \in \mathfrak{p}$). Similar results hold if $p = 2$.

3 Nonexistence Results

Theorem 3.1 *There are no almost-ordinary curves (curves with 2-rank $\sigma = g - 1$) in $\mathcal{H}_{g,2}$ for any g.*

Proof. Let X be a curve in $\mathcal{H}_{g,2}$ which is almost-ordinary. It follows from Theorem 2.1 that one of its $\mathbb{Z}/2\mathbb{Z}$ quotients must be almost-ordinary and the other two must be ordinary. Let C_1 and C_2 be the two quotients which are ordinary so that C_1 (resp. C_2) is defined by the equation $y^2 + y = f_1(x)$ (resp. $f_2(x)$) where f_1 (resp. f_2) only has simple poles. Then $f_1 + f_2$ must also have only simple poles and therefore the curve $C_{1,2}$, which is defined by $y^2 + y = f_1(x) + f_2(x)$, must also be ordinary. This gives a contradiction.

In some cases it happens that a given 2-rank can occur for curves of some type in $\mathcal{H}_{g,2}$ but not for curves of another type, as the following theorems indicate.

Theorem 3.2 *There exist curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ of 2-rank zero only if the two largest entries of $\mathfrak{p}$ are identical. (ie. the elements of $\mathfrak{p}$ can be put into order $g_1 = g_2 \geq g_3$)*

Proof. Let $X \in \mathcal{H}_{g,2}$ be a curve whose 2-rank equals zero. Then we can conclude from Theorem 2.1 that all three of the hyperelliptic quotients have 2-rank zero and therefore they can each be defined by $y^2 + y = f_i(x)$ where each f_i has a single pole. It follows that (at least) two of these three functions must have a pole of the same order, and therefore that (at least) two of the associated subgenera must be identical.

We define a partition $\mathfrak{p}$ to be unbalanced if it contains an element which is at least $\frac{g}{2}$. In particular, unbalanced partitions are of the form $\{\frac{g}{2}, *, *\}$ or $\{\frac{g+1}{2}, *, *\}$ depending on the parity of g.

Theorem 3.3 *There exist smooth curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ with 2-rank equal to one if and only if g is odd and $\mathfrak{p}$ is unbalanced. In particular, if g is even then there are no curves in $\mathcal{H}_{g,2}$ with 2-rank equal to one.*

Proof. Let $X \in \mathcal{H}_{g,2}$ be a curve whose 2-rank equals 1. Then two of the hyperelliptic quotients must have 2-rank zero while the third has 2-rank one. It follows without loss of generality that f_1 has a pole of order a at one point and f_2 has a pole of order b at another point where a and b are both odd. In that case we can compute that the curve X is of type $\{\frac{a-1}{2}, \frac{b-1}{2}, \frac{a+b}{2}\}$ which in turn implies that the genus of the curve X is $a+b-1$ (and is thus odd) while the genus of the curve C_3 is $\frac{a+b}{2} = \frac{g+1}{2}$.

A quite different but similarly restrictive results holds if we look at curves with 2-rank equal to 2.

Theorem 3.4 *Let $\mathfrak{p} = \{g_1, g_1, g_1\}$ be a totally balanced partition. Then there do not exist curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ with 2-rank equal to two.*

Proof. Let $X \in \mathcal{H}_{g,2}$ be a curve whose 2-rank is equal to 2. Let C_1, C_2, and C_3 be the three quotient curves and let σ_i be the 2-rank of C_i. Then it follows from Theorem 2.1 that (without loss of generality) either $\sigma_1 = 2$ and $\sigma_2 = \sigma_3 = 0$ or $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = 0$. However, the first case cannot happen, because it would imply that f_1 would have 3 poles while each of f_2 and f_3 would have a single pole.

Therefore we must be in the second case, in which f_1 and f_2 each have two poles and f_3 has one pole. It follows that we can assume that f_1 and f_2 each have poles at zero and infinity and that f_3 has a pole only at infinity. Without loss of generality, we may assume that $ord_\infty(f_1) \geq ord_\infty(f_3)$ which will in turn imply that $g_1 > g_3$. Therefore, $\mathfrak{p}$ cannot be a totally balanced partition.

Theorem 3.5 *Let $\mathfrak{p}$ be an unbalanced partition. Then there exist smooth curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ with 2-rank equal to σ only if $g \equiv \sigma$ (mod 2).*

Proof. Assume that X is a curve in $\mathcal{H}_{g,2,\mathfrak{p}}$ and that $\frac{g+1}{2} \in \mathfrak{p}$, so that g must be odd. Then there exists an involution $\tau \in Aut(X)$ such that the genus of $C_1 = X/ < \tau >$ is equal to $\frac{g+1}{2}$. It follows from the Riemann-Hurwitz formula that this cover must be etale. Therefore, if we apply the Deuring-Shafarevich formula we see that $\sigma_X = 2\sigma_{C_1} - 1$ is odd.

Similarly, if we assume that X is a curve in $\mathcal{H}_{g,2,\mathfrak{p}}$ and that $\frac{g}{2} \in \mathfrak{p}$ (and hence g is even), we note that it follows from the Riemann-Hurwitz formula that the cover $X \to C_1$ must be ramified at a single point. Again, it will follow from Deuring-Shafarevich that $\sigma_X = 2\sigma_{C_1}$ must be even.

Therefore, in both cases where we look at curves whose associated partitions are unbalanced we see that $\sigma_X \equiv g_X$ (mod 2). The converse will follow from results in Section 4

4 Existence Results

The main result in this section is that all 2-ranks occur for all types of curves in $\mathcal{H}_{g,2}$ except for those that we have proven do not occur in the previous section. In particular, we will prove the following theorem.

Theorem 4.1 *There exist curves of 2-rank σ in $\mathcal{H}_{g,2,\mathfrak{p}}$ for all $0 \leq \sigma \leq g$ except in the following cases:*

i. $\sigma = 0$, $\mathfrak{p} \neq \{g_1, g_1, g_3\}$ *with* $g_3 \leq g_1$.

ii. $\sigma = 1$, $\frac{g+1}{2} \notin \mathfrak{p}$.

iii. $\sigma = 2$, $\mathfrak{p}$ *totally balanced.*

iv. $\sigma = g - 1$.

v. $\sigma \not\equiv g$ *(mod 2),* $\mathfrak{p}$ *unbalanced.*

We will prove this theorem by induction on σ after looking at some base cases. Throughout these, let α be one of the elements of $\mathbb{F}_4$ other than one or zero.

Lemma 4.2 *Let $\mathfrak{p}$ be a partition which is neither completely balanced or, if g is odd, unbalanced. Then there exist curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ of 2-rank equal to two.*

Proof. Let $\mathfrak{p} = \{g_1, g_2, g_3\}$ with $g_1 \geq g_2 \geq g_3$. Let $a = 2g_3 + 1$, $b = 2(g_1 - g_3) - 1$ and $c = 2(g_2 + g_3 - g_1) + 1$. It is clear that a, b, and c are all odd, and that $a \geq c$. Furthermore, $b \geq 1$ because $\mathfrak{p}$ is not completely balanced and $c \geq 1$ because $g_1 \leq g/2$. If we now let $f_1 = x^a + \frac{1}{x^b}$ and $f_2 = \alpha x^c + \frac{1}{x^b}$ we see that $f_3 = f_1 + f_2 = x^a + \alpha x^c$ and a simple computation shows that the fibre product X will have 2-rank equal to 2 and lie in $\mathcal{H}_{g,2,\mathfrak{p}}$.

Lemma 4.3 *If* $\sigma = 3$ *then Theorem 4.1 holds. In particular, there are curves of* 2*-rank equal to three in each* $\mathcal{H}_{g,2,\mathfrak{p}}$ *except the case where* g *is even and* $\frac{g}{2} \in \mathfrak{p}$*.*

Proof. We consider two different cases. First, if g is odd and $\mathfrak{p}$ is unbalanced then $\frac{g+1}{2} \in \mathfrak{p}$ and in particular we may assume that the branch loci of f_1 and f_2 will be disjoint. If we let $f_1 = x^a + \frac{1}{x-1}$ and $f_2 = \frac{1}{x^b}$ then we will have $\sigma_1 = 1, \sigma_2 = 0$, and $\sigma_3 = 2$ so that $\sigma_X = 3$. Furthermore, $g_1 = \frac{a+1}{2}$, $g_2 = \frac{b-1}{2}$, and $g_3 = \frac{a+b+2}{2} = g_1 + g_2 + 1 = \frac{g+1}{2}$. We can choose a and b in order to get any unbalanced partition of g that we desire.

For the other case, assume that $\mathfrak{p} = \{g_1, g_2, g_3\}$ with $1 \leq g_3 \leq g_2 \leq g_1 \leq \frac{g-1}{2}$. Set $a = 2(g_1 - g_3) + 1$, $b = 2g_3 - 1$ and $d = 2(g_2 + g_3 - g_1) - 1$. We note that our hypotheses imply that a, b, and d are all odd positive numbers with $b \geq d$. Now, let $f_1 = x^a + \frac{\alpha}{x^b}$, $f_2 = x^a + \frac{1}{x^d}$ and $f_3 = f_1 + f_2$. Then the curve defined by $y^2 + y = f_i$ will have genus g_i and 2-rank equal to one, and therefore X will be a curve of 2-rank equal to three in $\mathcal{H}_{g,2,\mathfrak{p}}$.

Lemma 4.4 *If* $\sigma = 4$ *then Theorem 4.1 holds. In particular, there are curves of* 2*-rank equal to four in each* $\mathcal{H}_{g,2,\mathfrak{p}}$ *except the case where* g *is odd and* $\frac{g+1}{2} \in \mathfrak{p}$*.*

Proof. Assume that $\mathfrak{p} = \{g_1, g_2, g_3\}$ where $g_1 > g_2 \geq g_3$. Let $a = 2g_2 - 1$, $b = 2(g_1 - g_2) - 1$, and $c = 2(g_2 + g_3 - g_1) + 1$. One can easily check that a, b, and c are all positive odd numbers (recall that the fact that $\frac{g+1}{2} \notin \mathfrak{p}$ implies that $g_1 \leq g_2 + g_3$) and furthermore that $a \geq c$. Let $f_1 = x^a + \frac{1}{x^b} + \frac{1}{x+1}$, $f_3 = x^c + \frac{1}{x^b}$, and $f_2 = f_1 + f_3$. Then the curve defined by the equation $y^2 + y = f_i(x)$ has genus g_i and the fibre product will have 2-rank $\sigma = 4$ as desired.

On the other hand, assume that $g_1 = g_2 \geq g_3 \geq 2$. In this case, let $a = 2g_1 - 1$ and $b = 2g_3 - 3$. Then it is clear that a and b are positive odd integers with $a > b$. If we define $f_1 = x^a + \frac{1}{x}$ and $f_3 = x^b + \frac{1}{x} + \frac{1}{x+1}$ we can see that the curves will have the desired properties.

It remains to consider the partitions $\mathfrak{p} = \{\frac{g}{2}, \frac{g}{2}, 0\}$ in the case where g is even and $\mathfrak{p} = \{\frac{g-1}{2}, \frac{g-1}{2}, 1\}$ if g is odd. For the former, we let $f_1 = x^{g-3} + \frac{1}{x} + \frac{1}{x+1}$ and $f_2 = \alpha x$ and $f_3 = f_1 + f_2$. Then $g_1 = g_3 = g/2$ and $g_2 = 0$ while $\sigma_1 = \sigma_3 = 2$ and $\sigma_2 = 0$. For the latter, we note that g must be odd and $g > \sigma + 1 = 5$, so we may assume that $g \geq 7$. Let $f_1 = x^{g-4} + \frac{1}{x} + \frac{1}{x+1}$ and $f_2 = \alpha x$. These equations define curves with the desired genera and 2-ranks.

Remark 4.5 Now that we have shown that the theorem is true for $\sigma \leq 4$ we are ready to consider the inductive step. The key idea is to notice that if there is a curve X in $\mathcal{H}_{g,2,\{g_1,g_2,g_3\}}$ with 2-rank equal to σ then there will be a curve $\tilde{X}$ in $\mathcal{H}_{g+3,2,\{g_1+1,g_2+1,g_3+1\}}$ with 2-rank equal to $\sigma + 3$. In particular, if the three hyperelliptic quotients of X are defined by the equations $y^2 + y = f_i(x)$, then without loss of generality we may assume that none of the f_i have poles at infinity. Then we define $\tilde{f}_1 = f_1 + x$, $\tilde{f}_2 = f_2 + \alpha x$ and $\tilde{f}_3 = f_3 + (\alpha + 1)x$. It is clear that $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$ and that the curve $\tilde{X}$ defined by the fibre product of $y^2 + y = \tilde{f}_1(x)$ and $y^2 + y = \tilde{f}_2(x)$ will lie in $\mathcal{H}_{g+3,2,\{g_1+1,g_2+1,g_3+1\}}$ and have 2-rank $\sigma + 3$. Therefore, once we have shown that Theorem 4.1 holds for σ we have *almost* shown that it will hold for $\sigma+3$. However, to be complete there are still a few cases we must consider.

Lemma 4.6 *The component* $\mathcal{H}_{g,2,\{g/2,g/2,0\}}$ *contains curves of every even* 2*-rank.*

Proof. In order to construct a curve in $\mathcal{H}_{g,2,\{g/2,g/2,0\}}$ with 2-rank equal to $2k$ we first note that we can find a hyperelliptic curve of genus $g/2$ with 2-rank equal to k. Let us assume that this curve is defined by the equation $y^2+y=f_1(x)$ where f_1 has a pole at infinity. Let f_2 be some constant multiple of x so that $f_3=f_1+f_2$ will have the same poles (with the same orders) as f_1. Then it follows from our construction that the curve X will have 2-rank $2k$ and will lie in $\mathcal{H}_{g,2,\{g/2,g/2,0\}}$.

Lemma 4.7 *Let g be odd and $\mathfrak{p}$ be an unbalanced partition of g. Then $\mathcal{H}_{g,2,\mathfrak{p}}$ contains curves of all odd p-ranks* $\sigma=2k+1$.

Proof. If g is odd and $\mathfrak{p}$ is unbalanced, then $\mathfrak{p}=\{\frac{g+1}{2},g_1,g_2\}$ where $g_1\geq g_2$. We note that we can construct hyperelliptic curves C_1 and C_2 so that the genus of C_i is g_i and the 2-rank of C_i is k_i for all $0\leq k_i\leq g_i$. Furthermore, we can assume that the branch loci are distinct. If we let X be the fibre product of C_1 and C_2 and consider the third hyperelliptic quotient of X we see that it will have genus g_1+g_2+1 and 2-rank k_1+k_2+1. If we choose k_1 and k_2 so that $k_1+k_2=k$ then X will have 2-rank equal to σ and lie in $\mathcal{H}_{g,2,\mathfrak{p}}$.

Lemma 4.8 *Let g be odd and $\frac{g-1}{2}\in\mathfrak{p}$ but $\frac{g+1}{2}\notin\mathfrak{p}$. Then there are curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ with* 2*-rank equal to $2k$ for all $0\leq k\leq\frac{g-3}{2}$.*

Proof. Let $\mathfrak{p}=\{\frac{g-1}{2},g_1,g_2\}$ with $g_1\geq g_2>0$ and let $\sigma=2k$ be as above. Because $\sigma\leq g-3$ we have that $k\leq g_1+g_2-2$ and therefore we can choose k_1 and k_2 so that $k_1+k_2=k$ but $k_i<g_i$. In particular, we can define a function $h_1(x)$ which has k_1 poles (none of which are at infinity) so that if we look at $\frac{1}{2}\sum(n_i+1)$ where the sum runs over the poles of h_1, each of which is of order n_i, then we can get any integer which is at least k_1 and in particular we can get g_1-1. To be precise, we can choose h_1 so that the curve C_1 defined by $y^2+y=x^3+h_1(x)$ will have genus g_1 and 2-rank k_1. Similarly, we can choose h_2 with poles distinct from those of h_1 so that the curve C_2 defined by $y^2+y=\alpha x^3+h_2(x)$ will have genus g_2 and 2-rank k_2.

If we look at the normalization of the fibre product of C_1 and C_2 we see that the third quotient will be defined by the equation $y^2+y=(\alpha+1)x^3+h_1(x)+h_2(x)$ and therefore will have genus g_1+g_2-1 and 2-rank $k_1+k_2=k$. Thus, the curve X lies in $\mathcal{H}_{g,2,\{\frac{g-1}{2},g_1,g_2\}}$ and has 2-rank equal to $2k$, as desired.

Before we prove the main theorem in general, we look at the case where $\sigma=5$, in which we need to fill an extra gap.

Lemma 4.9 *If $\sigma=5$ then Theorem 4.1 holds. In particular, if $g\geq 7$ there are curves of* 2*-rank equal to five in each $\mathcal{H}_{g,2,\mathfrak{p}}$ except the case where g is even and $\frac{g}{2}\in\mathfrak{p}$.*

Proof. Let $\mathfrak{p} = \{g_1, g_2, g_3\}$ be a partition of g with $0 \leq g_3 \leq g_2 \leq g_1 \leq \frac{g+1}{2}$. We wish to show that there are curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ of 2-rank equal to five unless $g_1 = \frac{g}{2}$ (in which case g will be even). If $g_1 = \frac{g+1}{2}$ then the result follows from Lemma 4.7.

If $g_1 \leq \frac{g-1}{2}$ then it follows that $g_3 > 0$ and thus $\hat{\mathfrak{p}} = \{g_1 - 1, g_2 - 1, g_3 - 1\}$ gives a partition of $g - 3$ all of whose entries are at most $\frac{g-3}{2}$. Thus, by Lemma 4.2 there are curves in $\mathcal{H}_{g-3,2,\hat{p}}$ of 2-rank equal to 2 unless $\hat{p}$ (and therefore $\mathfrak{p}$ is completely balanced). By the induction argument in Remark 4.5 we therefore have curves whose 2-rank is equal to five in $\mathcal{H}_{g,2,\mathfrak{p}}$.

It remains to consider the case where $\mathfrak{p}$ is totally balanced: that is, where $g_1 = g_2 = g_3 = a$. To deal with this case, let $f_1 = x^a + \frac{1}{x^a}$ and $f_2 = x^a + \frac{1}{(x-1)^{a-2}} + \frac{1}{x-\alpha}$ and $f_3 = f_1 + f_2$. One can easily compute that these choices will lead to a curve $X\ in \mathcal{H}_{g,2,\{a,a,a\}}$ whose 2-rank is equal to 5.

We are finally ready to prove Theorem 4.1.

Proof. Given the results of the above lemmata, it suffices to consider the case where $\sigma \geq 6$. In this case, we only need to prove that there are curves of 2-rank equal to σ in every partition if $g \equiv \sigma \pmod 2$ and that there are curves of 2 rank equal to σ in every partition whose entries are all at most $\frac{g-1}{2}$ if $g \not\equiv \sigma \pmod 2$.

If $0 \in \mathfrak{p}$ then $\mathfrak{p}$ must be unbalanced, and therefore we only need to consider the case where $g \equiv \sigma \pmod 2$. The result then follows from Lemma 4.7 if g is odd and from Lemma 4.6 if g is even. Similarly, if $\frac{g+1}{2} \in \mathfrak{p}$ the result follows from Lemma 4.7.

Next we consider the case where $\mathfrak{p} = \{\frac{g}{2}, g_1, g_2\}$ with g_1 and g_2 both positive. It follows from Theorem 3.5 that there are no curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ whose 2-rank is odd, so we wish to show that there will be curves of all even 2-ranks. We note that $\hat{p} = \{\frac{g}{2} - 1, g_1 - 1, g_2 - 1\}$ gives a partition of $g - 3$. If $g \equiv \sigma \pmod 2$ then $g - 3 \equiv \sigma - 3$ and there will be curves in $\mathcal{H}_{g-3,2,\hat{\mathfrak{p}}}$ of 2-rank $\sigma - 3$ by the inductive hypotheses which can be used to construct curves in $\mathcal{H}_{g,2,\mathfrak{p}}$ of 2-rank equal to σ by the inductive argument in 4.5.

If $\mathfrak{p} = \{\frac{g-1}{2}, g_1, g_2\}$ we note that both g_1 and g_2 must be positive. It follows from Lemma 4.8 that there are curves of every even 2-rank less than $g-1$ in $\mathcal{H}_{g,2,\mathfrak{p}}$. To construct the curves of odd 2-rank σ, we note that $\hat{\mathfrak{p}} = \{\frac{g-3}{2}, g_1 - 1, g_2 - 1\}$ gives a partition of $g-3$, and that $g - 3 \equiv \sigma - 3 \pmod 2$ and therefore there are curves in $\mathcal{H}_{g-3,2,\hat{p}}$ of 2-rank equal to $\sigma - 3$. The result then follows from the inductive process described in Remark 4.5.

If all entries of $\mathfrak{p}$ are at least 1 and at most $\frac{g-2}{2}$, we note $\hat{\mathfrak{p}} = \{g_1 - 1, g_2 - 1, g_3 - 1\}$ gives a partition of $\hat{g} = g - 3$ such that each $\hat{g}_i = g_i - 1$ is at most $\frac{\hat{g}-1}{2}$ and therefore there exist curves of 2-rank $\sigma - 3$ in $\mathcal{H}_{\hat{g},2,\hat{\mathfrak{p}}}$. By the inductive procedure described in Remark 4.5 we can construct a curve in $\mathcal{H}_{g,2,\mathfrak{p}}$ with 2-rank equal to σ, proving the theorem.

In [9], Zhu proves that there exist hyperelliptic curves with no extra automorphisms of every possible 2-rank. The following result shows that there are often hyperelliptic curves that *do* have an extra involution.

Corollary 4.10 *There are hyperelliptic curves of genus g and* 2*-rank* σ *which contain an additional involution in their automorphism group if and only if* $g \equiv \sigma$ *(mod* 2*).*

Proof. It is well known that the hyperelliptic involution is contained in the center of the automorphism group of a curve. Therefore, if there is another involution in the automorphism group then we must have a Klein-four action on the curve and therefore we will be in the setup above. Furthermore, it follows that the partition $\mathfrak{p}$ corresponding to this curve contains a zero and is therefore either $\mathfrak{p} = \{\frac{g+1}{2}, \frac{g-1}{2}, 0\}$ or $\mathfrak{p} = \{g/2, g/2, 0\}$. In either case, the partition is unbalanced and therefore $g \equiv \sigma$ (mod 2) by Theorem 3.5.

Conversely, it follows from Theorem 4.1 that if $g \equiv \sigma$ (mod 2) then there will exist curves in this partition, which will therefore be both hyperelliptic and contain an extra involution.

We note that this does not answer the question of the automorphism groups fully, as the curves may have automorphisms of degree greater than two. We examine the question of the possible 2-ranks of hyperelliptic curves with extra automorphisms in depth in [1].

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