Math Faculty Publications

Math

1-2014

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Abstract

In this note, we consider ordered partitions of integers such that each entry is no more than a fixed portion of the sum. We give a method for constructing all such compositions as well as both an explicit formula and a generating function describing the number of k-tuples whose entries are bounded in this way and sum to a fixed value g.

COMPOSITIONS OF INTEGERS WITH BOUNDED PARTS

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Abstract

In this note, we consider ordered partitions of integers such that each entry is no more than a fixed portion of the sum. We give a method for constructing all such compositions as well as both an explicit formula and a generating function describing the number of k-tuples whose entries are bounded in this way and sum to a fixed value g.

1. Introduction

Imagine having 100 pieces of candy to split among three children, but with the following restriction: the oldest child cannot receive more than one-third of the candy, the middle child cannot get more than two-fifths of the candy, and the youngest child cannot get more than two-sevenths of the candy. How many ways are there to split the candy between the three children? Are there more or fewer ways to split 101 pieces with the same restrictions? In this note, we address these questions and the more general question of counting compositions (also known as ordered partitions) of integers so that no part is more than a fixed portion of the total.

More precisely, let us fix an integer k and for each $1 \le i \le k$ let α_i be a rational number so that the sum of any $k-1$ of the α_i is at most 1 but that all k of the α_i add up to more than 1 (note that if the α_i sum to 1 or less than one then the question is trivial). We wish to count the number of ordered k-tuples of integers $[g_1, g_2, \ldots, g_k]$ with $0 \le g_i \le \alpha_i \sum_{j=1}^{k} g_j$ for each i. We will call such a composition α-*communal* and our goal is to understand the structure of the set of α-communal compositions.

As is often the case with counting questions, one might reasonably ask for either an explicit formula for $f(g)$, the number of α-communal compositions of a given integer g, or for a nice closed form of the generating function defined by $F(x) = \sum f(g)x^g$. In this note we answer both questions: in particular, Section 2 describes an explicit formula for $f(g)$. In Section 3 we show that the set of α-communal compositions forms a monoid and we describe an explicit structure for this monoid.

These results lead to Theorem 3.3, in which we give a closed form description of $F(x)$.

We close our note with a section giving examples of the results in the previous sections. One special case we consider is the case where all of the α_i are equal to $\frac{1}{k-1}$. This situation was studied by the author in [5], where we referred to the k-tuples simply as communal compositions. Example 4.2 shows that some of the results of that paper are special cases of the results in this note.

It is worth pointing out that the problem we are considering in this note is related, but not identical, to the "17 Horses Puzzle," in which three sons are supposed to split 17 horses so that one son gets $\frac{1}{2}$, one son gets $\frac{1}{3}$ and the final son gets $\frac{1}{9}$ of the horses. A fuller discussion of this problem attributing it to Tartaglia in the sixteenth century can be found in [6, Problem 2.11].

2. Combinatorial Formula

In this section, we give an explicit formula for $f(g)$, the number of α-communal k-tuples summing to g, for any fixed integer g. In particular, let $\alpha = \{\alpha_i\}_{i=1}^k$ be a set of rational numbers so that the sum of any $k-1$ of these numbers is at most 1 but the sum of all k is at least 1. We wish to count the number of ordered k-tuples of integers $[g_1, \ldots, g_k]$ so that $\sum g_i = g$ and $0 \le g_i \le \alpha_i g$ for all i.

Given an α-communal k-tuple $[g_1, \ldots, g_k]$ with $\sum g_i = g$, we set $\epsilon_i = \lfloor \alpha_i g \rfloor - g_i$. The condition that our k-tuple is α-communal implies that each $\epsilon_i \in \mathbb{Z}_{\ge 0}$. We set $s_g = \sum_i \epsilon_i$ and note that it is a simple calculation to see that $s_g = \sum_i \lfloor \alpha_i g \rfloor - g$.

Conversely, given a set of k nonnegative integers $\{\epsilon_i\}$ so that $\sum \epsilon_i = s_g$, we set $g_i = \lfloor \alpha_i g \rfloor - \epsilon_i$. It is clear that $g_i \le \lfloor \alpha_i g \rfloor$. On the other hand, our hypotheses on the α_i include the fact that for any fixed j we have that $\sum_{i \ne j} \alpha_i \le 1$. Thus, we compute:

$$
\begin{aligned}
\sum_{i \ne j} \alpha_i &\le 1 \\
\sum_{i \ne j} \alpha_i g &\le g \\
\sum_{i \ne j} \lfloor \alpha_i g \rfloor &\le g \\
\sum_i \lfloor \alpha_i g \rfloor - g &\le \lfloor \alpha_j g \rfloor \\
s_g &\le \lfloor \alpha_j g \rfloor,
\end{aligned}
$$

which implies that $\epsilon_j \leq \lfloor \alpha_i g \rfloor$ and therefore that each $g_j \geq 0$. Moreover, one can check that $\sum g_i = g$, implying that $[g_1, \ldots, g_k]$ is an α-communal k-tuple summing to g.

In particular, there is a natural bijection between the set of α-communal k-tuples summing to g and the number of k-tuples of nonnegative integers summing to s_g. In order to count these, we will use the following lemma, which is standard in combinatorial number theory. One proof can be found in [3, Proposition 21.5].

Lemma 2.1. *The number of solutions to the equation $x_1 + \ldots + x_k = m$, where all of the x_i are nonnegative integers, is equal to the binomial coefficient $\binom{m+k-1}{k-1}$.*

The following theorem is an immediate consequence.

Theorem 2.2. *The number of α-communal k-tuples whose entries sum to g is given by the binomial coefficient*

$$f(g) = \left(\begin{array}{c} \sum_{i=1}^{k} \lfloor \alpha_i g \rfloor - g + k - 1 \\ k - 1 \end{array} \right).$$

If the rational number α_i is expressed in lowest terms as $\frac{m_i}{n_i}$ and we set n to be the least common multiple of the n_i, then we note that the formula in Theorem 2.2 can be instead expressed as a collection of n polynomials of degree $k-1$ depending on the value of $g \bmod n$. We return to this formula in explicit examples in Section 4.

3. Structure of α-Communal Compositions

To begin, let us fix a k-tuple $(\alpha_1, \ldots, \alpha_k)$. We leave the proof of the following lemma to the reader:

Lemma 3.1. *If $\mathbf{x} = [x_1, \ldots, x_k]$ and $\mathbf{y} = [y_1, \ldots, y_k]$ are α-communal k-tuples then so is their sum $\mathbf{x} + \mathbf{y} = [x_1 + y_1, \ldots, x_k + y_k]$.*

In particular, the set of α-communal k-tuples forms a submonoid of the additive monoid $\mathbb{Z}_{\geq 0}^k$. This leads to the natural question of finding a set of generators for the set. In order to do so, let us first introduce some notation. We write the rational number α_i as the fraction $\frac{m_i}{n_i}$ in lowest terms and define N to be the product $\prod_{i=1}^{k} n_i$. Additionally, we set $A = N(\sum_{i=1}^{k} \alpha_i - 1)$ and $\hat{\alpha}_i = 1 - \sum_{j \neq i} \alpha_j$. For each i, let us define the k-tuple $\mathbf{x}_i$ as follows:

$$\mathbf{x}_i = \frac{N}{n_i}[\alpha_1, \ldots, \alpha_{i-1}, \hat{\alpha}_i, \alpha_{i+1}, \ldots, \alpha_k].$$

We note that the entries of each $\mathbf{x}_i$ are nonnegative integers because of the assumption that the sum of any $k - 1$ of the α_i is at most 1. Moreover, it is an

easy exercise to check that each $\mathbf{x}_i$ is α-communal, and by Lemma 3.1 every triple obtained as a nonnegative integral linear combination of the $\mathbf{x}_i$ will be as well.

Lemma 3.2. *Let* $\mathfrak{g} = [g_1, \ldots, g_k]$ *be an* α-*communal* k-*tuple with* $g = \sum g_i$. *Then one can write* $\mathfrak{g}$ *as the sum of the* $\mathbf{x}_i$ *in the following way:*

$$\mathfrak{g} = \sum_{j=1}^{k} \frac{m_j g - n_j g_j}{A} \mathbf{x}_j.$$

Proof. Let $\mathfrak{h} = (h_1, \ldots, h_k)$ be the k-tuple defined by $\mathfrak{h} = \sum_{j=1}^{k} (m_j g - n_j g_j) \mathbf{x}_j$. In particular, we can compute that the ith coordinate of $\mathfrak{h}$ is:

$$
\begin{aligned}
h_i &= \sum_{j \neq i} (m_j g - n_j g_j) \frac{N m_j}{n_i n_j} + (m_i g - n_i g_i) \left(\frac{N}{n_i} - \sum_{j \neq i} \frac{N m_j}{n_i n_j} \right) \\
&= \alpha_i N g - \sum_{j \neq i} \alpha_i N g_j - N g_i + \sum_{j \neq i} \alpha_j N g_i \\
&= N g_i \left(\sum_{j=1}^{k} \alpha_j - 1 \right) \\
&= A g_i \ .
\end{aligned}
$$

The lemma immediately follows. $\square$

By assumption, $m_j g \geq n_j g_j$ for each j, so the coefficients are all nonnegative. In particular, if each $m_j g - n_j g_j$ is a multiple of A then we have shown that one can write the $\mathfrak{g}$ as an integral combination of the $\mathbf{x}_i$. In particular, if $A = 1$ then the $\mathbf{x}_i$ form a basis for the monoid of α-communal k-tuples, a situation which we explore in Example 4.1. In the case where $A > 1$ we will not get all k-tuples in this manner. To cover this case, let a_j be the least residue of $m_j g - n_j g_j \mod A$. Then it follows that $(m_j g - n_j g_j - a_j)/A$ is a nonnegative integer and we compute:

$$\sum_{i=1}^{k} \frac{m_i g - n_i g_i - a_i}{A} \mathbf{x}_i = [g_1, \ldots, g_k] - \sum_{i=1}^{k} \frac{a_i}{A} \mathbf{x}_i = [g_1, \ldots, g_k] - [b_1, \ldots, b_k]$$

where

$$b_j = \frac{1}{A} \left(N \hat{\alpha}_j \frac{a_j}{n_j} + N \alpha_j \sum_{i \neq j} \frac{a_i}{n_i} \right).$$

In particular, we can write any α-communal k-tuple $\mathbf{g} = [g_1, \ldots, g_k]$ in a unique way as the sum of a 'base' k-tuple $[b_1, \ldots, b_k]$ and a nonnegative integral combination of the $\mathbf{x}_i$. Moreover, because the k-tuples $\mathbf{g}$ and $\mathbf{x}_i$ consist of integers, it must be

the case that the b_i are all integers, and therefore the base k-tuples that we need to consider are exactly those arising from k-tuples $(a_1, \ldots, a_k)$ of least residues which make them integers. In particular, they will be the k-tuples in the set:

$$\mathcal{A} = \left\{ (a_1, \ldots, a_k) \middle| \begin{array}{l} 0 \le a_i < A \\ N\hat{\alpha}_j \frac{a_j}{n_j} + N\alpha_j \sum_{i \ne j} \frac{a_i}{n_i} \equiv 0 \bmod A \text{ for all } 1 \le j \le k \end{array} \right\}$$

which will have at most A^k elements and for 'generic' choices of the α_i will have A^{k-1} elements – if the m_i and n_i are all relatively prime to A then one deduces that the congruence conditions are in fact equivalent and allows one to get an explicit formula for a_k in terms of the other a_i.

For each $\mathfrak{a} \in \mathcal{A}$, we define $b(\mathfrak{a})$ to be the sum of the entries in the corresponding base k-tuple, and we can compute:

$$\begin{aligned}
b(\mathfrak{a}) = \sum_{j=1}^k b_j &= \sum_{j=1}^k \frac{N}{A} \left(\hat{\alpha}_j \frac{a_j}{n_j} + \alpha_j \sum_{i \ne j} \frac{a_i}{n_i} \right) \\
&= \frac{N}{A} \sum_{j=1}^k \hat{\alpha}_j \frac{a_j}{n_j} + \frac{N}{A} \sum_{i \ne j} \frac{\alpha_j a_i}{n_i} \\
&= \frac{N}{A} \left(\sum_{i=1}^k \frac{a_i}{n_i} \right).
\end{aligned}$$

It follows that the number of α-communal k-tuples summing to g is the same as the number of ways to write g as the sum of a number of the form $b(\mathfrak{a})$ for some k-tuple $(a_1, \ldots, a_k) \in \mathcal{A}$ and a nonnegative integral linear combination of the numbers $\frac{N}{n_i}$. Theorem 3.3 is an immediate consequence using basic facts on generating functions (see [2] or [7], for example).

Theorem 3.3. *Let $f(g)$ be the number of k-tuples of nonnegative integers $g_1, \ldots, g_k$ so that $\sum g_i = g$ and $g_i \le \alpha_i g$, where the α_i are rational numbers as described above. Then the function $f(g)$ can be described by a generating function in the following way:*

$$F(x) = \sum_{g=0}^\infty f(g) x^g = \frac{\sum_{\mathfrak{a} \in \mathcal{A}} x^{b(\mathfrak{a})}}{\prod_{i=1}^k (1 - x^{N/n_i})}.$$

4. Examples

Computing explicit formulas from Theorem 3.3 requires an understanding of the set $\mathcal{A}$, which depends strongly on the relationships between the α_i. This can be complicated in general, but in many specific cases, such as when the m_i and n_i

share common factors, the terms reduce greatly and one can write simple closed forms for the generating function $F(x)$. We close this note with some examples and applications.

Example 4.1. Let $k = 3$ with $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{5}$. In particular, $\alpha_1 + \alpha_2 + \alpha_3 = \frac{31}{30}$ so we see that $A = 1$. As discussed in the previous section, this implies that every α-communal triple can be written as a nonnegative integral combination of the triples $\mathbf{x}_1 = [7, 5, 3]$, $\mathbf{x}_2 = [5, 3, 2]$, $\mathbf{x}_3 = [3, 2, 1]$. In particular, the generating function whose coefficients give us the number of α-communal triples summing to g is given by $F(x) = ((1 - x^{15})(1 - x^{10})(1 - x^6))^{-1}$. At the same time, Theorem 2.2 tells us that a formula for the number of α-communal triples summing to g is given by

$$f(g) = \binom{\lfloor \frac{g}{2} \rfloor + \lfloor \frac{g}{3} \rfloor + \lfloor \frac{g}{5} \rfloor - g + 2}{2}.$$

Other examples of k-tuples so that the term A equals 1, making it particularly easy to write down the structure of α-communal compositions, include $\alpha = (\frac{2}{5}, \frac{1}{17}, \frac{7}{23}, \frac{9}{38})$, $(\frac{1}{4}, \frac{1}{7}, \frac{2}{11}, \frac{4}{13}, \frac{2}{17})$, and $(\frac{3}{11}, \frac{3}{13}, \frac{2}{15}, \frac{1}{17}, \frac{7}{23})$.

Example 4.2. We next wish to apply our results to the classical problem of counting triangles with a fixed perimeter and integer sides, as considered by Andrews in [1]. In our context, this is the case where $k = 3$ and each of the $\alpha_i = \frac{1}{2}$, so each $m_i = 1$ and each $n_i = 2$. We wish to consider the more general situation where we set $\alpha_i = \frac{1}{k-1}$ for each $1 \le i \le k$.

Let ℓ be the least residue of g modulo $k - 1$. In particular we have $\lfloor \frac{g}{k-1} \rfloor = \frac{g - \ell}{k-1} \in \mathbb{Z}$. It follows from Theorem 2.2 that the number of α-communal k-tuples summing to g is given by the function

$$f(g) = \binom{\frac{g - \ell k}{k-1} + k - 1}{k - 1}.$$

As an illustration, in the case $k = 3$ this reduces to the formula $f(g) = \frac{1}{8}(g^2 + 6g + 8)$ if g is even and $\frac{1}{8}(g^2 - 1)$ if g is odd.

If we instead wish to find the generating function describing the sequence $\{f(g)\}$, we note that in the notation of Section 3 we can compute that $A = (k - 1)^{k-1}$ and the set $\mathcal{A}$ consists of all k-tuples $(a_1, \ldots a_k)$ so that $0 \le a_i < (k - 1)^{k-1}$ and all of the a_i are congruent modulo $k - 1$. In particular, we can write the set $\mathcal{A}$ as a disjoint union of sets $\mathcal{A}_s$ for $0 \le s \le (k - 2)$ where all of the a_i are congruent to s modulo $k - 1$. We note that if $\mathfrak{a} \in \mathcal{A}_s$ then we have that $b(\mathfrak{a}) = \sum a_i \equiv s \pmod{(k - 1)}$ as well.

We observe that for any $g \equiv 0 \pmod{(k-1)}$, the number of ways to write it as a sum of k numbers which are multiples of $k - 1$ is straightforward to compute, and standard results about generating functions imply that

$$\sum_{\mathfrak{a} \in \mathcal{A}_0} x^{\sum a_i} = (1 + x^{k-1} + x^{2(k-1)} + \ldots + x^{(k-2)(k-1)})^k.$$

Similarly, there is a bijection between k-tuples in $\mathcal{A}_s$ whose entries sum to g and k-tuples in $\mathcal{A}_0$ whose entries sum to $g - ks$, allowing one to compute that

$$\sum_{\mathfrak{a} \in \mathcal{A}_s} x^{b(\mathfrak{a})} = x^{ks}(1 + x^{k-1} + \ldots + x^{(k-2)(k-1)})^k.$$

We now use Theorem 3.3 to compute the generating function explicitly:

$$
\begin{aligned}
F(x) &= \frac{\sum_{\mathfrak{a} \in \mathcal{A}} x^{b(\mathfrak{a})}}{\prod_{i=1}^{k}(1 - x^{N/n_i})} \\
&= \frac{(1 + x^k + \ldots + x^{k(k-2)})(1 + x^{k-1} + x^{2(k-1)} + \ldots + x^{(k-2)(k-1)})^k}{(1 - x^{(k-1)^2})^k} \\
&= \frac{(1 - x^{k(k-1)})(1 - x^{(k-1)^2})^k}{(1 - x^k)(1 - x^{k-1})^k(1 - x^{(k-1)^2})^k} \\
&= \frac{1 - x^{k(k-1)}}{(1 - x^k)(1 - x^{k-1})^k} ,
\end{aligned}
$$

which agrees with the formula given in [5, Thm 10].

Example 4.3. Let $k = 3$ and $m_i = 1$, and $n_1 = n_2 = 2$ with $n_3 = n \geq 2$. In this case, $A = 4$ and one can see that $(a_1, a_2, a_3) \in \mathcal{A}$ if and only if $a_1 \equiv a_2 \pmod 2$ and $a_3 \equiv a_1 + \frac{a_1 + a_2}{2}n \pmod 2$. In particular, if n is odd then $\mathcal{A}$ consists of the sixteen triples:

$$
\begin{array}{cccc}
(0,0,0) & (0,0,2) & (2,2,0) & (2,2,2) \\
(0,2,1) & (0,2,3) & (2,0,1) & (2,0,3) \\
(1,1,0) & (1,1,2) & (3,3,0) & (3,3,2) \\
(1,3,1) & (1,3,3) & (3,1,1) & (3,1,3)
\end{array} .
$$

One can then use Theorem 3.3 to compute that the generating function in this case is:

$$
\begin{aligned}
F(x) &= \frac{1 + x^2 + x^n + 2x^{n+1} + x^{n+2} + 2x^{n+3} + x^{2n} + 2x^{2n+1} + x^{2n+2} + 2x^{2n+3} + x^{3n} + x^{3n+2}}{(1 - x^4)(1 - x^{2n})^2} \\
&= \frac{1 + 2x^{n+1} + x^{2n}}{(1 - x^2)(1 - x^n)(1 - x^{2n})} .
\end{aligned}
$$

Similarly, if n is even one can show that the generating function simplifies to

$$F(x) = \frac{1 + x^{n+1}}{(1 - x^2)(1 - x^n)^2} .$$

One can additionally use Theorem 2.2 to show that the number of α-communal triples summing to g is given by $f(g) = \frac{1}{2}(\lfloor \frac{g}{n} \rfloor + 1)(\lfloor \frac{g}{n} \rfloor + \epsilon_g)$, where $\epsilon_g = 2$ if g is even and $\epsilon_g = 0$ if g is odd.

It is worth noting that this question was one of the original motivations for this note, as it is related to the question of counting the irreducible components of the moduli space of dihedral covers of the projective line. While we will not go into details here, the interested reader might consult [5, §5] and [4] for the similar problem comparing $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$-communal triples to the moduli space of $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\mathbb{P}^1$.

Example 4.4. Returning to the example from the opening paragraph of this note, let us let $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{5}$ and $\alpha_3 = \frac{2}{7}$. Then $A = 2$ and $\mathcal{A}$ consists of triples (a_1, a_2, a_3) with either one or three entries equalling 0 and the others equalling 1. In particular, we see that any division of candy that satisfies our restrictions can be written as the sum of one of the triples in the set $\{[0, 0, 0], [6, 7, 5], [8, 10, 7], [9, 11, 8]\}$ and an integral linear combination of the triples $[5, 6, 4], [7, 8, 6], [11, 14, 10]$. Moreover, the generating function associated to this problem is

$$F(x) = \frac{1 + x^{18} + x^{25} + x^{28}}{(1 - x^{15})(1 - x^{21})(1 - x^{35})}.$$

A computer algebra system will now tell us that the expansion of this as a power series includes the terms

$$F(x) = \ldots x^{97} + 3x^{98} + 3x^{99} + 3x^{100} + x^{101} + 3x^{102} + 3x^{103} + \ldots$$

which tells us that there are three ways to distribute 100 pieces of candy according to these rules (in particular, they are $[32, 40, 28], [33, 39, 28]$, and $[33, 40, 27]$) but a unique way of dividing 101 pieces, $[33, 40, 28]$.

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