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Abstract

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Keywords

sandpiles, chip-firing, rotor-router model, ribbon graphs, planarity

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SANDPILES, SPANNING TREES, AND PLANE DUALITY*

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Abstract. Let G be a connected, loopless multigraph. The sandpile group of G is a finite abelian group associated to G whose order is equal to the number of spanning trees in G. Holroyd et al. used a dynamical process on graphs called rotor-routing to define a simply transitive action of the sandpile group of G on its set of spanning trees. Their definition depends on two pieces of auxiliary data: a choice of a ribbon graph structure on G, and a choice of a root vertex. Chan, Church, and Grochow showed that if G is a planar ribbon graph, it has a canonical rotor-routing action associated to it; i.e., the rotor-routing action is actually independent of the choice of root vertex. It is well known that the spanning trees of a planar graph G are in canonical bijection with those of its planar dual G^*, and furthermore that the sandpile groups of G and G^* are isomorphic. Thus, one can ask: are the two rotor-routing actions, of the sandpile group of G on its spanning trees, and of the sandpile group of G^* on its spanning trees, compatible under plane duality? In this paper, we give an affirmative answer to this question, which had been conjectured by Baker.

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1. Introduction.

Let G be a connected multigraph with no loop edges. The *sandpile group* of G is a finite abelian group whose order is equal to the number of spanning trees in G; it is the group of *degree zero divisors* of G modulo the equivalence relation generated by *lending moves*. (We will recall all relevant definitions in section 2.)

In [8], Holroyd et al. use a dynamical process on graphs called *rotor-routing* to define a simply transitive action of the sandpile group of G on its set of spanning trees. Rotor-routing itself was introduced in [9] under the name "Eulerian walkers" and has been rediscovered several times in different fields: see [8] for a concise history of the topic.

The definition of the rotor-routing action on G given in [8] involves two pieces of auxiliary data. First, the action is defined with respect to a choice of a root vertex $v \in V(G)$, or *basepoint*. Second, it depends on a *ribbon graph* structure on G: a choice of a cyclic ordering of the set of edges incident to each vertex v. Note that such a choice of cyclic orders defines an embedding of G on some closed, oriented surface S, in which all cyclic orders correspond to a positive orientation, say with respect to S.

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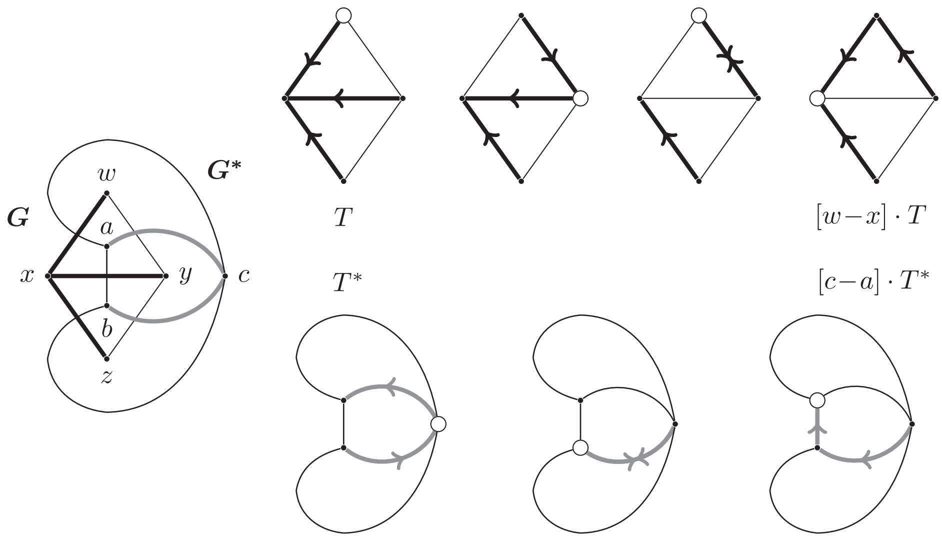

FIG. 1. *This figure shows the result of applying the element* $[w-x]$ *of* $\mathcal{S}(G)$ *to the spanning tree* T *and, in the bottom row, the result of applying the element* $[c-a]$ *of* $\mathcal{S}(G^*)$ *to* T^**. The graph* G *has all rotors oriented clockwise relative to the page, and its planar dual* G^* *has all rotors oriented counterclockwise. We chose* x *and* a *as our basepoints of* G *and* G^* *for the respective computations. The isomorphism* $\mathcal{S}(G) \cong \mathcal{S}(G^*)$ *identifies* $[w-x]$ *and* $[c-a]$*, so the trees* $[w-x] \cdot T$ *and* $[c-a] \cdot T^*$ *must be dual trees, as shown on the right.*

We say that G is a *planar* ribbon graph if S is just a sphere, i.e., if the chosen ribbon structure equips G with an embedding into the plane.

A recent paper of Chan, Church, and Grochow [5] answers a question of Ellenberg [7] by proving that the rotor-routing action does not depend on the choice of basepoint if and only if G is a planar ribbon graph. This result is somewhat surprising, and as a nice consequence of it, we may henceforth refer to *the* rotor-routing action on a planar ribbon graph, without further reference to a choice of basepoint.

Any graph G embedded in the plane has a planar dual graph G^* whose spanning trees are in canonical bijection with those of G. Moreover, the sandpile groups of G and G^* are, up to sign, canonically isomorphic [1] (see also [6]). Thus, one would hope that the two rotor-routing actions, of the sandpile group of G on the set $\mathcal{T}(G)$ of its spanning trees, and of the sandpile group of G^* on its spanning trees, are compatible.

This was, in fact, exactly the conjecture suggested to us by Baker. In this paper, we provide a proof of Baker's conjecture on the compatibility of the rotor-routing action of the sandpile group with plane duality. See Theorem 3.1 for the precise statement, and see Figure 1 for an example illustrating the result.

We begin with preliminary definitions of the sandpile group and rotor-routing in section 2. The proof of our main result occupies section 3. The key idea of our proof is the *angle* between two spanning trees T and T' of G: see Definition 3.3. The angle from T to T' remembers the element of the sandpile group that takes T to T' under rotor-routing. On the other hand, we are able to show, using a direct geometric argument, that the angle is compatible with plane duality, so the main theorem follows.

We would also like to refer the reader to the recent preprint [3], which arrives at another proof of Theorem 3.1 via a completely different route. In that manuscript, Baker and Wang prove that the bijections obtained by Bernardi in [4, Theorem 45] give rise to another simply transitive action of the sandpile group on the spanning trees of a ribbon graph G with a fixed root vertex. They show that this action is compatible with plane duality and that it coincides with the rotor-routing action when G is planar. It would be interesting to study the relationship between these two approaches further.

2. Preliminaries.

2.1. The sandpile group. Let $G = (V, E)$ be a finite connected loopless multigraph with vertex set V and edge multiset E. The set of *divisors* on G is the free abelian group on the vertices: $\mathrm{Div}(G) = \mathbb{Z}V$. We imagine a divisor $D = \sum_{v \in V} a_v\, v$ to be an assignment of $D(v) := a_v$ chips to each vertex v, keeping in mind that this number may be negative. We write $\mathrm{Div}^0(G)$ for the subgroup of divisors whose net number of chips $\sum D(v)$ is zero.

A *lending move* by a vertex v consists of removing $\deg(v)$ chips from v and distributing them along incident edges to the vertices neighboring v. In other words, letting $n(v, w)$ denote the number of edges between v and w, a lending move by v performed on a divisor D produces a divisor D' given by

$$D'(w) = \begin{cases} D(w) + n(v, w) & \text{if } w \neq v, \\ D(v) - \deg(v) & \text{if } w = v. \end{cases}$$

Notice that lending moves do not change the total number of chips in a divisor. Divisors D and D' are *linearly equivalent*, denoted $D \sim D'$, if one can be obtained from the other by a sequence of lending moves at various vertices. The *sandpile group* of G is

$$\mathcal{S}(G) = \mathrm{Div}^0(G)/\sim .$$

The sandpile group of a graph is also variously known as the *Jacobian* of G, the *Picard group* $\mathrm{Pic}^0(G)$, or the *critical group* of G.

2.2. Integral cuts and cycles. Fix an arbitrary orientation on the edges E, and let $\mathbb{Z}E$ be the free abelian group on these oriented edges. If $e = \{u, v\} \in E$ is given the orientation (u, v), we write $e^+ = \mathrm{head}(e) = v$ and $e^- = \mathrm{tail}(e) = u$. We identify $-e$ with the oppositely oriented edge (v, u). Each directed cycle on the underlying undirected graph G may be thought of as an element of $\mathbb{Z}E$, and the $\mathbb{Z}$-linear span of these cycles in $\mathbb{Z}E$ is the *integral cycle space* for G, which we denote by $\mathcal{C}$.

Next, for any subset $U \subset V$, the collection of all edges joining a vertex of U to a vertex of $V \setminus U$ is called a *cut*. By directing all of these edges from vertices in U to vertices in $V \setminus U$, we can identify this cut with an element of $\mathbb{Z}E$. If U consists of a single vertex v, this cut is called a *vertex cut* at v. The integer span of all cuts is the *integral cut space* for G and is denoted by $\mathcal{C}^*$. Note that the vertex cuts generate the cut space.

Define

$$\mathcal{E}(G) = \mathbb{Z}E/(\mathcal{C} + \mathcal{C}^*).$$

We now identify $\mathcal{E}(G)$ with the sandpile group $\mathcal{S}(G)$, as follows. Define the *boundary map* $\mathbb{Z}E \to \mathrm{Div}^0(G)$ by sending each edge e to $e^+ - e^-$. The boundary map is surjective since G is connected, and its kernel is exactly the cycle space of G, so it identifies $\mathrm{Div}^0(G)$ with $\mathbb{Z}E/\mathcal{C}$. Now, given $D \in \mathrm{Div}^0(G)$, let D_v be the boundary of a vertex cut at the vertex v. Then $D + D_v$ is the divisor obtained from D by performing a lending move at v. Therefore the boundary map induces an isomorphism,

$$\partial_G\colon\ \mathcal{E}(G) \overset{\cong}{\to} \mathcal{S}(G), \qquad e \mapsto [e^+ - e^-],$$

as was proved in [1, Proposition 8]. We will sometimes write ∂ instead of ∂_G for brevity.

2.3. Rotor-routing action on spanning trees. Fix a *ribbon graph* structure on G; i.e., for each vertex v, fix a cyclic ordering of the edges incident to v. Fix a vertex q. A *rotor configuration with basepoint* q is the choice for each vertex $v \neq q$ of an edge, $\rho(v)$, incident to v. We orient each edge $\rho(v)$ so that its tail is v.

Let D be a divisor on G, thought of as a chip configuration on G, and let ρ be a rotor configuration with basepoint q. We now recall the *rotor-routing* process, by which a divisor D transforms ρ into a new rotor configuration ρ'. *Firing* a vertex v consists of updating ρ by replacing $\rho(v)$ with the next edge in the cyclic ordering of edges at v, then removing a chip from v and placing it at the other end of the new edge $\rho(v)$. Note that firing v a total of $\deg(v)$ times does not change the original rotor configuration but transforms D by a lending move at v. Now, every divisor D on G is linearly equivalent to a divisor D' with $D'(v) \geq 0$ for all $v \neq q$; see, e.g., [2, Proposition 3.1]. From that point, [8] shows that, solely through vertex firings, all chips may be routed into q, and the rotor configuration at the end of this process depends solely on the divisor class of D.

Let $\mathcal{T}(G)$ denote the set of spanning trees of G. Rooting $T \in \mathcal{T}(G)$ at q uniquely determines a rotor configuration ρ_T: for each vertex $v \neq q$, set $\rho_T(v)$ to be the edge incident to v on the path in T from v to q. Given a divisor class $[D] \in \mathcal{S}(G)$, use the rotor-routing process to route all chips into q (at which point, all chips will be gone since $\deg(D) = 0$). It is shown in [8] that the resulting rotor configuration is a spanning tree, directed into q. Call the underlying undirected spanning tree $[D] \cdot T$. Then according to [8] the resulting map,

$$\mu_G\colon\ \mathcal{S}(G) \times \mathcal{T}(G) \to \mathcal{T}(G), \qquad ([D], T) \mapsto [D] \cdot T,$$

is a simply transitive action of $\mathcal{S}(G)$ on $\mathcal{T}(G)$.

2.4. Planar duality. Now suppose that $G = (V, E)$ is a planar ribbon graph, and let $G^* = (V^*, E^*)$ be its *planar dual graph*, whose vertices are the faces of G and whose edges cross the edges of G. We shall assume throughout that both G and G^* are loopless; i.e., G has neither bridges nor loops. We write e^* for the edge of G^* crossing the edge e of G. Each spanning tree of G determines a spanning tree of G^*: namely, there is a natural bijection

$$\delta\colon\ \mathcal{T}(G) \xrightarrow{\cong} \mathcal{T}(G^*)$$

sending T to the tree $T^* = \{e^* \in E^* : e \in E \setminus T\}$.

Let us call the orientation of the plane that agrees with the cyclic orderings of G *clockwise*. Then we fix once and for all the following *planar dual ribbon graph structure* on G^*: take the cyclic orderings of the edges at the vertices of G^* to be *counterclockwise* with respect to the plane.

In order to define $\mathbb{Z}E$, we fixed an arbitrary orientation of the edges of G. To define $\mathbb{Z}E^*$, we will now choose a compatible orientation on the edges of G^*. For an oriented edge e of G, let e' (respectively, e'') denote the edge at $v = e^-$ before (respectively, after) e in the cyclic order at v. Now, call the face between e' and e at v the face *before* e, and call the face between e and e'' at v the face *after* e. Then we orient e^* so that its head is the face of G before e and its tail is the face of G after e. For example, in Figure 1, with the rotors of G oriented clockwise relative to the page, suppose that e is the directed edge from x to y. Then e^* is the directed edge in G^* from b to a.

Since directed cycles of G are directed cuts of G^* and vice versa, mapping each edge to its dual produces an isomorphism $\mathcal{E}(G) \cong \mathcal{E}(G^*)$, and hence we get an isomorphism ϕ of sandpile groups labeled as in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}(G) & \xrightarrow{\cong} & \mathcal{E}(G^*) \\ \partial_G \downarrow & & \downarrow \partial_{G^*} \\ \mathcal{S}(G) & \xrightarrow{\phi} & \mathcal{S}(G^*). \end{array}$$

3. Compatibility of rotor-routing with duality. Let G be any planar ribbon graph such that both G and its dual G^* are loopless. In the previous section, we established an isomorphism $\phi\colon \mathcal{S}(G) \to \mathcal{S}(G^*)$ that depended on a single global choice of orientation of the E^* derived from the orientation E. With respect to this choice, we may now state the main theorem of the paper, as follows.

THEOREM 3.1. *The diagram*

$$\begin{array}{ccc} \mathcal{S}(G) \times \mathcal{T}(G) & \xrightarrow{\mu_G} & \mathcal{T}(G) \\ \phi\times\delta \downarrow & & \downarrow \delta \\ \mathcal{S}(G^*) \times \mathcal{T}(G^*) & \xrightarrow{\mu_{G^*}} & \mathcal{T}(G^*) \end{array}$$

commutes. In other words, the rotor-routing action is compatible with plane duality.

In the rest of this section, we prove Theorem 3.1. We begin with a topological definition of the angle between two spanning trees; this definition applies to all ribbon graphs, not just planar ones, and is the key idea in our proof of Theorem 3.1.

Suppose that G is any ribbon graph, and let e and e' be directed edges emanating from a vertex u. Suppose that in the cyclic order starting from $e = e_0$, the edges between e and e' are $e_0, e_1, \dots, e_k$, where $e_k = e'$, all directed outward from u. Define the *angle* between e and e' at u by

$$\angle^u(e, e') = \sum_{i=1}^{k} \partial e_i \in \mathcal{S}(G).$$

Recall that ∂ denotes the boundary map sending a directed edge e to the element $[e^+ - e^-] \in \mathcal{S}(G)$. Note that the sum includes e' but not e. See Figure 2.

LEMMA 3.2. *Suppose G is a planar ribbon graph, and let $e_0, \dots, e_k$ be consecutive outgoing edges from some vertex u in the cyclic order at u. For $i = 0, \dots, k$, let r_i be the face of G (equivalently, the vertex of G^*), lying to the right of e_i (with respect to the cyclic order at u). Then*

$$\phi(\angle^u(e_0, e_k)) = [r_0 - r_k] \in \mathcal{S}(G^*).$$

Proof. We have $\phi(\partial e_i) = [r_{i-1} - r_i]$, so by linearity $\phi(\angle^u(e_0, e_k))$ is the telescoping sum $[(r_0 - r_1) + (r_1 - r_2) + \cdots + (r_{k-1} - r_k)]$, proving the claim. ∎

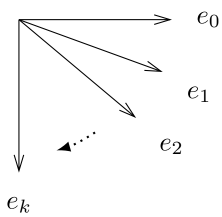

FIG. 2. $\angle^u(e_0, e_k) = \partial e_1 + \cdots + \partial e_k$.

DEFINITION 3.3. *Let G be an arbitrary ribbon graph, and let T and T' be two spanning trees of G. Let $v \in V$ be any vertex. As in section 2.3, let ρ_T and $\rho_{T'}$ be the rotor configurations based at v arising from orienting T and T' towards v.*

The angle *between T and T' based at v, denoted $\angle_v(T, T')$, is the sum of the angles between their edges at each nonroot vertex. That is,*

$$\angle_v(T, T') := \sum_{u \in V \setminus \{v\}} \angle^u(\rho_T(u), \rho_{T'}(u)) \in \mathcal{S}(G).$$

LEMMA 3.4. *Let G be any ribbon graph, and let T be a spanning tree of G. For any vertex v and any $[D] \in \mathcal{S}(G)$ we have*

$$\angle_v(T, [D] \cdot T)) = [-D].$$

Here, the rotor-routing action of $[D]$ on T is computed with respect to the basepoint v.

Proof. Without loss of generality, we may choose D to be a chip configuration that is nonnegative at vertices other than v. Consider the rotor-routing process that calculates $[D] \cdot T$. We will say that the directed edge (x, y) is *activated* if a chip is sent from vertex x to vertex y during this process. Note that, when the chip is fired, the chip configuration on the graph changes by $\partial(x, y) = y - x$. Since at the end of the rotor-routing process there are no chips left on the graph, it follows that

$$[D] + \sum_e \partial e = 0,$$

where the sum is over the multiset of edges that have been activated during the process.

Next, we claim that the angle between T and $[D] \cdot T$ is in fact equal to $\sum_e \partial e$, where the sum is again over the multiset of activated edges. This is because at each vertex $u \neq v$, the sum of the boundaries of all outgoing edges e at u is $0 \in \mathcal{S}(G)$; after all, this sum corresponds to a lending move at u. So the sum over all activated edges leaving u is exactly the angle at u between the edge of T leaving u and that of T', and the claim follows. Summarizing, we have

$$\angle_v(T, [D] \cdot T)) = \sum_e \partial e = [-D]. \qquad \square$$

COROLLARY 3.5. *Let G be any planar ribbon graph, and let T and T' be spanning trees of G rooted at the same vertex v. Then $\angle_v(T, T') = 0$ if and only if $T = T'$.*

Proof. Assume that $\angle_v(T, T') = 0$, and let $[D] \in \mathcal{S}(G)$ take T to T' under the rotor-routing action with basepoint v. It follows from [8, Lemma 3.17] that the element $[D]$ exists and is unique. Then by Lemma 3.4, $[D] = 0$, so $T = T'$. The converse is clear. ∎

Remark 3.6. It follows from Lemma 3.4 and from [5, Theorem 2] that the notion of angle between trees for G is independent of the choice of root vertex for the trees if and only if G is a planar ribbon graph. Indeed, Lemma 3.4 shows that $\angle_v(T, T')$ is exactly the element of $\mathcal{S}(G)$ sending T' to T in the rotor-routing action based at v, and the rotor-routing action is basepoint-independent if and only if G is a planar ribbon graph by [5]. Thus, if G is planar, we will henceforth write $\angle(T, T')$ for the angle between T and T', computed with respect to any vertex.

We can now prove our main lemma.

LEMMA 3.7. *Let G be a planar ribbon graph, and let T and T' be spanning trees of G. Then*

$$\phi(\angle(T, T')) = \angle(T^*, T'^*)\,.$$

Proof. Given a spanning tree T and an edge e not in T, we call the unique cycle $C(e)$ in $T \cup \{e\}$ the *fundamental cycle* of e with respect to T. We first note that there is a sequence of trees $T = T_0, T_1, \ldots, T_r = T'$ such that for each j the trees T_{j+1} and T_j have exactly $n-1$ edges in common. If $T = T'$, this statement is trivially true. Otherwise, pick $e' \in T' \setminus T$; then the fundamental cycle of e' with respect to T must contain some edge $e \in T \setminus T'$. Set $T_1 = T \cup \{e'\} \setminus \{e\}$. Then T_1 and T' have smaller symmetric difference, so repeating, we produce a sequence of spanning trees as desired. It follows by induction that we may assume $T' = T \cup \{e'\} \setminus \{e\}$.

In fact, we may further assume, again by induction, that e and e' are edges incident to a common face of G. Indeed, since $T^* \cup \{e^*\} \setminus \{e'^*\} = T'^*$ is acyclic, the fundamental cycle $C(e^*)$ of e^* with respect to T^* contains e'^*. Now starting at e^* and proceeding along the cycle $C(e^*)$ in either direction, let $e^* = e_0^*, e_1^*, \ldots, e_s^* = e'^*$ be the sequence of edges traversed. Then

$$T^*,\ (T^* \cup \{e^*\})\backslash\{e_1^*\},\ (T^* \cup \{e^*\})\backslash\{e_2^*\}, \ldots,\ (T^* \cup \{e^*\})\backslash\{e_s^*\}$$

is a sequence of trees in G^* such that the symmetric difference of any consecutive pair of trees consists of two edges of G^* adjacent to the same vertex. Now passing to G, we conclude that

$$T,\ (T \cup \{e_1\})\backslash\{e\},\ (T \cup \{e_2\})\backslash\{e\}, \ldots,\ (T \cup \{e'\})\backslash\{e\}$$

is a sequence of trees in G such that the symmetric difference of any consecutive pair of trees consists of two edges of G incident to the same face.

Thus, from here on, we assume that $T' = (T \cup \{e'\}) \setminus \{e\}$, where $e, e' \in E(G)$ are incident to a common face, which we call f. Write $e = xy$ and $e' = x'y'$ for vertices x, y, x', y' of $V(G)$ such that f is to the left of the edge e when it is traversed in the direction $x \to y$, and f is to the right of the edge e' when it is traversed in the direction $x' \to y'$. Write $\mathcal{C}$ for the fundamental cycle in $T \cup \{e'\}$; it is illustrated in Figure 3. (Here and throughout the rest of the proof, we assume a clockwise orientation on the rotors of G simply in order to talk about the left and right sides of an edge freely. For example, the face to the right of an oriented edge $e = (x, y)$ should be interpreted as the face coming in between e and the edge after e in the cyclic order at x.)

By Remark 3.6, the calculation of the angle $\angle(T, T') \in \mathcal{S}(G)$ is independent of the choice of root vertex. Choose x' as the root, and orient T and T' towards x'. We wish to study the sum of the angles at each vertex $v \neq x'$ of G between the edges of T and T' that are outgoing from v.

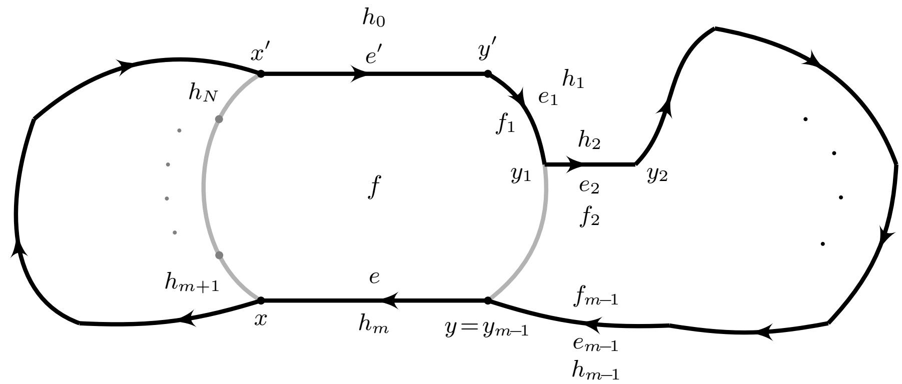

FIG. 3. *The fundamental cycle $\mathcal{C}$ of $T \cup \{e'\}$, shaded in black.*

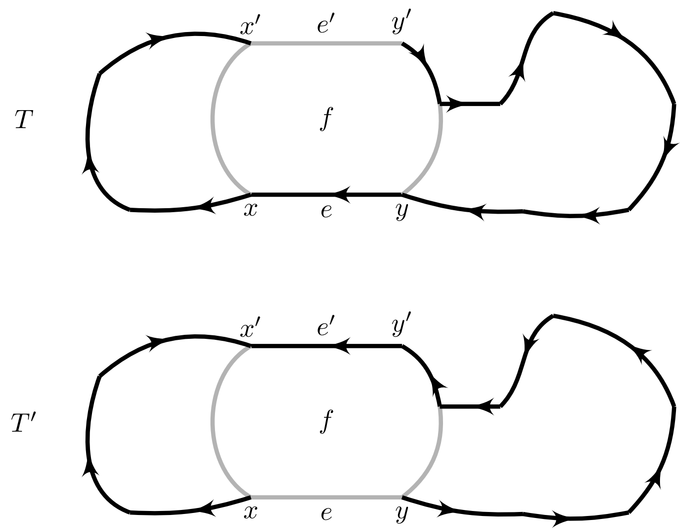

FIG. 4. *Parts of the trees T and T', rooted at the vertex x'.*

Having rooted the trees at x', we start by observing that the path between y and y' in T is directed from y' to y, whereas in T' it has the opposite orientation. This is illustrated in Figure 4. Furthermore, all other edges shared by T and T' have the same orientation. Indeed, consider a vertex v not on $\mathcal{C}$, and say its unique path in T to x' first meets $\mathcal{C}$ at v'; then the same path v–v' in T' must be an initial subpath of the unique path in T' from v to x', so in particular the edge leaving v is unchanged.

Let us fix some notation before going further. Write

$$y' = y_0, e_1, y_1, e_2, \ldots, y_{m-1} = y$$

for the sequence of vertices and directed edges in the y'–y path in T. For each directed edge e_i, we write f_i (respectively, h_i) for the face of G to the right (respectively, left) of e_i.

For convenience, we extend the notation above as follows. We denote by h_0 the face of G to the left of e' when oriented from x' to y', and we denote by h_m the face

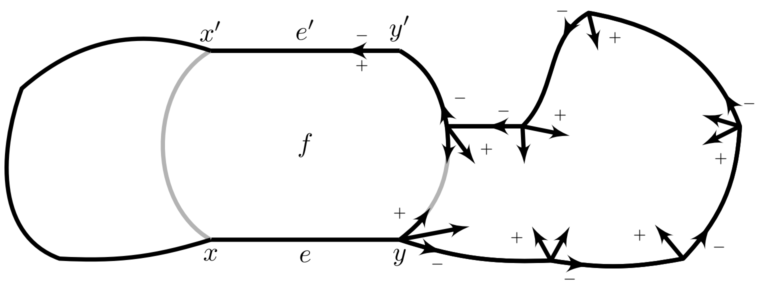

FIG. 5. *$\angle(T, T') \in \mathcal{S}(G)$ and $\phi(\angle(T, T')) \in \mathcal{S}(G^*)$, the former drawn with arrows, and the latter drawn with plus and minus signs.*

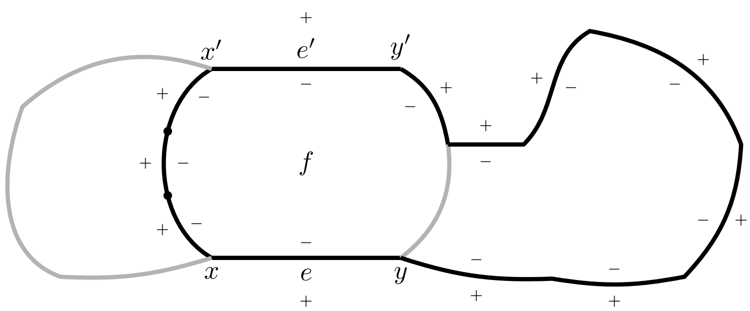

FIG. 6. *The cycle C in black, and $\partial_{G^*}(C^*)$.*

of G to the left of e when oriented from y to x. Next, consider the path from x to x' that bounds f and such that f lies on its right. Call the faces on the *left* side of this x–x' path $h_{m+1}, \ldots, h_N$. See Figure 3.

Letting $e_0 = e'$ and $e_m = e$, the angle between T and T' then is given by

$$\angle(T, T') = \sum_{i=0}^{m-1} \angle^{y_i}(e_{i+1}, e_i) \in \mathcal{S}(G),$$

where in each expression in the sum we regard each edge as being oriented away from y_i in turn. Then by Lemma 3.2, we have

$$\phi(\angle(T, T')) = (f_1 - h_0) + (f_2 - h_1) + \cdots + (f_{m-1} - h_{m-2}) + (f - h_{m-1}) \in \mathcal{S}(G^*).$$

The angle between T and T' is shown in Figure 5. The signs indicate $\phi(\angle(T, T')) \in \mathcal{S}(G^*)$.

Next, consider the oriented cycle C running from x' to y', then along edges of T from y' to x, then along edges of f back to x', as shown in Figure 6. The dual C^* of C is a cut of G^*, so $\partial_{G^*}(C^*) = 0 \in \mathcal{S}(G^*)$. On the other hand,

$$\partial_{G^*}(C^*) = (h_0 - f) + (h_1 - f_1) + \cdots + (h_{m-1} - f_{m-1}) + \sum_{i=m}^{N}(h_i - f).$$

The signs in Figure 6 indicate $\partial_{G^*}(C^*) \in \mathcal{S}(G^*)$.

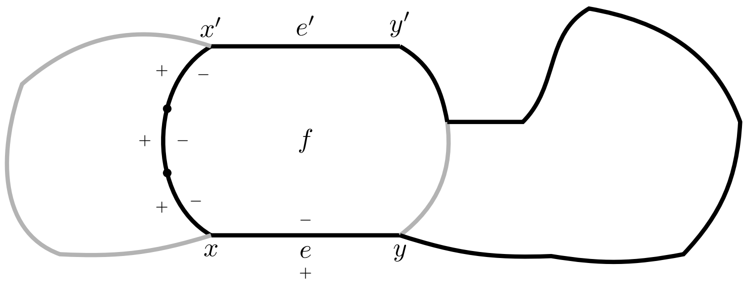

Fig. 7. $\phi(\angle(T, T')) + \partial_{G^*}(C^*) \in \mathcal{S}(G^*)$.

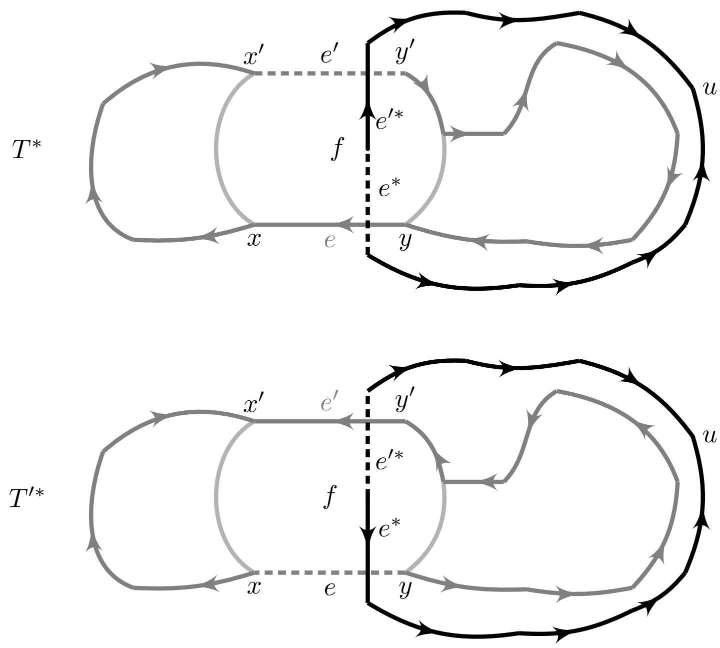

Fig. 8. *Parts of the trees T^* and T'^*, rooted at u.*

Summing, we have

$$\phi(\angle(T, T'))) + \partial_{G^*}(C^*) = \sum_{i=m}^{N}(h_i - f).$$

This sum is shown in Figure 7.

But this sum is exactly $\angle(T^*, T'^*)$. To see this, root the trees T^* and T'^* at a vertex u of G^* on the cycle in $T^* \cup \{e^*\}$ but different from f, as illustrated in Figure 8. Then the only nonzero vertex angle contributing to $\angle(T^*, T'^*)$ is the angle at the vertex f, and by definition this angle is $\sum_{i=m}^{N}(h_i - f)$, as shown in Figure 9. So we are done. $\square$

We now prove our main result.

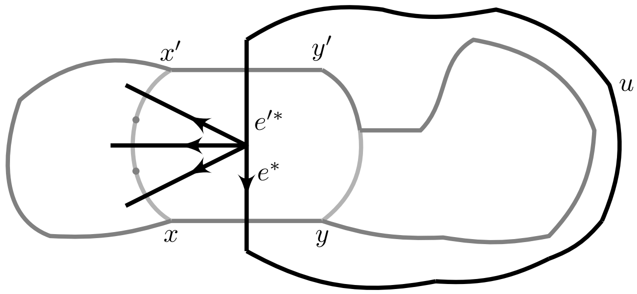

Fig. 9. $\angle(T^*, T'^*)$.

Proof of Theorem 3.1. Given $[D] \in \mathcal{S}(G)$ and $T \in \mathcal{T}(G)$, let $T' = [D] \cdot T$, and let $T'' = \phi([D]) \cdot T^*$. We would like to show that $T'' = T'^*$. By Lemma 3.4,

$$\phi(\angle(T, T')) = \phi([-D]) = \angle(T^*, T'').$$

By Lemma 3.7,

$$\phi(\angle(T, T')) = \angle(T^*, T'^*).$$

Hence, $\angle(T^*, T'') = \angle(T^*, T'^*)$. Therefore, $\angle(T'', T'^*) = 0$, and the result then follows from Corollary 3.5. ∎

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