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Abstract

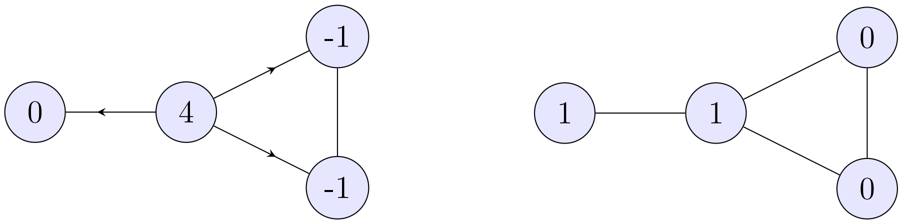
In this note, we describe a construction that leads to families of graphs whose critical groups are cyclic. For some of these families we are able to give a formula for the number of spanning trees of the graph, which then determines the group exactly.

1 Introduction and Background

This article will discuss some results related to the critical group of a finite connected graph G. While there are many ways to define the critical group, we will describe it in terms of a chip-firing game. In particular, we define a *configuration* on the graph G to be a function $\delta : V(G) \to \mathbb{Z}$, which we think of as assigning an integer number of chips to each vertex of G. Given a configuration, we define its *degree* to be the total number of chips assigned.

We next define transitions between configurations, by letting a *move* consist of choosing a vertex and either borrowing one chip from each adjacent vertex or firing one chip to each adjacent vertex. See Figure 1 for one example. We will say that two configurations are *equivalent* if one can get from one to the other through a sequence of these moves.

This setup may appear purely combinatorial in nature but it has a number of interesting applications in areas such as statistical physics, cryptography, algebraic geometry, and economics. We define the *critical group* of G to be the set of equivalence classes of configurations with degree zero. This set is naturally endowed with



(a) Before firing the center vertex (b) After firing the center vertex

Figure 1: Configurations on a graph before and after firing the center vertex

an abelian group structure where the group operation is addition of chips at corresponding vertices. We will denote this group by $K(G)$. Due to analogies with the set of divisors on an algebraic curve up to linear equivalence, this group is also known as the Jacobian of the graph G. For more details on these connections to algebraic geometry, we refer the reader to [4].

It is well-known that for a given graph on n vertices the critical group of G is isomorphic to $\mathbb{Z}^{n-1}/Im(\mathcal{L}^*)$, where $\mathcal{L}^*$ is the reduced Laplacian matrix of the graph G (see [2], [13] for details). One can compute the group structure of this quotient by computing the Smith Normal Form of the matrix $\mathcal{L}^*$. While efficient algorithms to do this are known, they often do not take into account the combinatorial structure of the graph. Several recent papers including [3], [6], and [8] attempt to use this structure in order to gain some insight into critical groups. Some of these results use the fact that the order of the critical group of a graph is equal to the number of spanning trees of that graph, which is a corollary of Kirchhoff's Matrix Tree Theorem. One result that is well known (see, for example, [6, Prop 1.2]) and which we will use repeatedly is the following:

Lemma 1.1. *Let G_1 and G_2 be two graphs and let H be the graph obtained by identifying a single vertex of G_1 with a single vertex of G_2. Then the critical group of H is isomorphic to the direct sum of the critical groups of G_1 and G_2.*

Given a graph G, it is natural to ask what the minimal number of elements needed to generate the critical group of G is. The extreme cases are handled by letting G be a tree, in which case the critical group is trivial, and letting G be the complete graph K_n, in which case the critical group is $(\mathbb{Z}/n\mathbb{Z})^{n-2}$. We also note that it follows from Lemma 1.1 that for any finite abelian group $\Gamma \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_r\mathbb{Z}$ it is possible to construct a graph G whose critical group is Γ by starting with r cycles of length $m_1, \ldots, m_r$ and identifying a single vertex on each of the cycles. While this construction shows that the rank of the critical group of a graph can be arbitrarily large, Wagner conjectured in [16, Conj 4.2] that the probability that a suitably defined random graph has a cyclic critical group approaches one. While this conjecture has recently been shown to be false, and Wood shows in [17, Cor 9.5] that the probability that a random graph has cyclic critical group is less than 0.8, there is still significant evidence that most random graphs have cyclic critical groups. In this note we will construct large families of graphs for which the critical group will be cyclic and we will discuss a method that can be used to compute the order of this cyclic group.

2 Adding Chains To Graphs

Given a graph G and two vertices $x, y \in V(G)$ we define $\delta_{x,y}$ to be the configuration on G so that $\delta_{x,y}(x) = 1, \delta_{x,y}(y) = -1$ and $\delta_{x,y}(v) = 0$ for $v \neq x, y$. We note that $\delta_{x,y} = -\delta_{y,x}$, and in particular the two divisors will generate the same subgroup of $K(G)$.

Definition 2.1. A *generating pair* of vertices for a graph G is a pair $\{x, y\} \subset V(G)$ so that the configuration $\delta_{x,y}$ is a generator of the critical group of G. Equivalently, $\{x, y\}$ will be a generating pair if any configuration of degree zero is equivalent to a configuration which has value zero except possibly at x and y.

Example 2.2. Let G be an n-cycle. More explicitly, let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$ and an edge between x_i and x_j if and only if $i \equiv j \pm 1 \bmod n$. Let δ be any configuration of total degree 0 on G. We claim that δ is equivalent to a multiple of δ_{x_{n-1},x_n}.

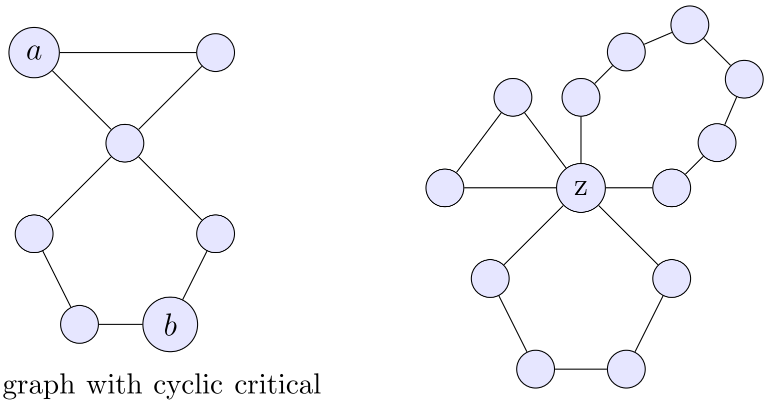
To see this, we let δ_1 be the configuration obtained from δ by borrowing $\delta(x_1)$ times at the vertex x_2. In particular, δ_1 will be the configuration defined by setting $\delta_1(x_1) = 0, \delta_1(x_2) = \delta(x_2) + 2\delta(x_1), \delta_1(x_3) = \delta(x_3) - \delta(x_1)$, and $\delta_1(x_i) = \delta(x_i)$ for all $i \geq 4$. For each $2 \leq k \leq n-2$ we define δ_k inductively as the configuration obtained from δ_{k-1} by borrowing $\delta_{k-1}(x_k)$ times at x_{k+1}.

We note that the configuration δ_{n-2} is equivalent to δ and $\delta_{n-2}(x_i) = 0$ except possibly at $i = n-1, n$. This verifies our claim and in particular proves that $\{x_{n-1}, x_n\}$ is a generating pair for G. More generally, one can show that the pair $\{x_i, x_j\}$ is a generating pair if and only if $gcd(i-j, n) = 1$.

It is not always the case that there is a generating pair consisting of two adjacent vertices. For example, if G is the graph in Figure 2a it follows from Lemma 1.1 that $K(G) \cong \mathbb{Z}/15\mathbb{Z}$ but that $\delta_{x,y}$ will either have order three or five for any pair of adjacent vertices. However, for the vertices labelled a and b one can see that $\delta_{a,b}$ will generate the full group.

We note that even in a situation where a graph has a cyclic critical group then there does not need to be a generating pair. The following example, provided by an anonymous referee, describes such a situation, answering a question posed by Lorenzini in [11, Remark 2.11].

Example 2.3. Let G be the graph in Figure 2b. By Lemma 1.1, $K(G) \cong \mathbb{Z}/105\mathbb{Z}$. Moreover, if z is the labelled vertex and $x \neq z$ is a different vertex on a cycle of size $d_x \in \{3, 5, 7\}$ then we note that the divisor $\delta_{x,z}$ has order d_x. For any two vertices x, y both of which are distinct from z, the divisor $\delta_{x,y}$ can be written as $\delta_{x,z} - \delta_{y,z}$, and therefore has order equal to $lcm(d_x, d_y) \in \{3, 5, 7, 15, 21, 35\}$ and in particular not equal to $|K(G)|$.



(a) A graph with cyclic critical group and no adjacent generating pairs

(b) A graph with cyclic critical group and no generating pairs

Figure 2: Examples

In the situation where our graph has a known generating pair, then we are able to construct a family of graphs which also have cyclic critical groups and known generating pairs due to the following theorem, which is the main result of this section.

Theorem 2.4. *Let x and y be a generating pair for G. Let $\tilde{G}$ be the graph G with an additional path of $\ell \geq 1$ edges (and $\ell - 1$ new vertices) between the vertices x and y. Then any pair of consecutive vertices along this path are a generating pair for $\tilde{G}$. In particular, $K(\tilde{G})$ is cyclic.*

Proof. Let G be a graph and $\{x, y\}$ be a generating pair for G. In particular, this means that for any configuration δ on G we can do a series of moves so that the resulting configuration has chips only on x and y.

Let $\tilde{G}$ be the graph with an additional path of length ℓ between vertices x and y. To be precise, $V(\tilde{G}) = V(G) \cup \{x_1, \dots, x_{\ell-1}\}$ and the edges of $\tilde{G}$ will be the edges of G along with edges connecting x_i and x_{i+1} for $1 \leq i \leq \ell - 2$ as well as edges connecting x to x_1 and $x_{\ell-1}$ to y. By convention, we set $x_0 = x$ and $x_\ell = y$.

Given a configuration $\tilde{\delta}$ on $\tilde{G}$ we can consider its restriction $\tilde{\delta}|_G$ as a configuration (not necessarily of degree zero) on G. We know there exists a sequence of legal moves that will make this configuration have chips only on the two vertices x and y. We perform this sequence of moves on $\tilde{\delta}$ and denote the resulting configuration on $\tilde{G}$ by δ_0.

We have now moved all of the chips in the configuration onto the chain connecting x and y, and we can therefore consolidate these on any two adjacent vertices. To be explicit, choose two adjacent vertices x_i and x_{i+1}. If $i \geq 1$ then for each $1 \leq j \leq i$ we let δ_j be the configuration obtained by borrowing $\delta_{j-1}(x_{j-1})$ times at the vertex x_j. In particular, the configuration δ_i will only have a nonzero value for vertices in $\{x_i, \dots, x_\ell\}$.

We continue by defining δ_j for $j > i$. In particular, for each $i < j \leq \ell - 1$ we let δ_j be the configuration obtained by borrowing $\delta_{j-1}(x_{\ell-j})$ times at the vertex $x_{\ell-j-1}$. At the end of this process, the resulting configuration $\delta_{\ell-1}$ will only have a nonzero number of chips on the vertices x_i and x_{i+1}. In particular, we have shown that every configuration on $\tilde{G}$ of degree zero is equivalent to a multiple of the divisor $\delta_{x_i,x_{i+1}}$ and therefore $\{x_i, x_{i+1}\}$ is a generating pair for $\tilde{G}$. □

We note that Theorem 2.4 is also a consequence of results in [9, Sect.2]. However, our proof is more elementary.

Example 2.5. Let G be the 'house' graph as pictured in Figure 3 with vertices as labelled. Assume that δ is a configuration of total degree zero on G. The fact that a 3-cycle has cyclic critical group and that any pair of adjacent vertices is a generating pair for the graph tells us that there is a sequence of moves that will lead to an equivalent divisor δ_1 with $\delta_1(z) = 0$. In particular, we can let δ_1 be the divisor obtained by borrowing $\delta(z)$ times at the vertex x.

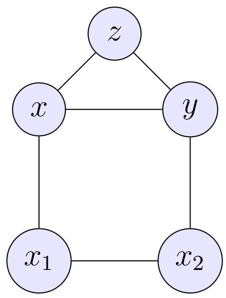


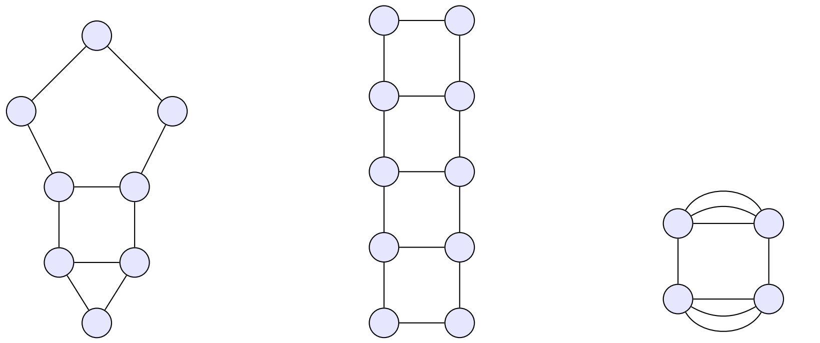
Figure 3: The one-story house is one simple example of a stack of polygons.

If we now let γ be the divisor obtained by borrowing $\delta_1(x)$ times at the vertex x_1 and $\delta_1(y)$ times at the vertex x_2, we can check that $\gamma(v)$ is only nonzero at x_1, x_2. In particular, (x_1, x_2) is a generating pair for G. In a similar manner, we could show that (x, x_1) and (x_2, y) are also generating pairs for G.

One can generalize the construction in Example 2.5 to more general stacks of polygons. In particular, let $(k_1, \ldots, k_n)$ be a sequence of integers with each $k_i \geq 2$. Define the graph G_1 to be a k_1-cycle and, for each $1 < i \leq n$ define the graph G_i by starting with graph G_{i-1} and adding a path of $k_i - 1$ edges between any two consecutive vertices of the path added at the previous step. The resulting graph G_n will consist of a stack of polygons with $k_1, \ldots, k_n$ sides. One example is that the stack corresponding to $(3, 4)$ or $(4, 3)$ are isomorphic to the house graph in Example 2.5. See Figure 4 for additional examples. It follows from inductive applications of Theorem 2.4 that $K(G_n)$ is cyclic; we note that similar results are discussed in [12].

We conclude this section by discussing some similarities between our result and results of Dino Lorenzini. In particular, [10, Thm 5.1] gives the following result:

Theorem 2.6. *Let G be a connected graph with vertices x, y so that there are $c > 0$ edges which have both x and y as their endpoints. Moreover, let G_1 be the graph obtained by deleting all edges between the two vertices x and y. If $|K(G)|$ and $|K(G_1)|$ are relatively prime then $K(G)$ is cyclic.*



(a) $(5, 4, 3)$ (b) $(4, 4, 4, 4)$ (c) $(2, 2, 4, 2, 2)$

Figure 4: Polygonal stacks corresponding to $(k_1, \ldots, k_n)$

In [12], he gives an alternate proof of this theorem and strengthens the result somewhat. In particular, he is able to prove:

Theorem 2.7. *Let G be a connected graph with vertices x, y connected by at least one edge so that $|K(G)|$ and $|K(G_1)|$ are relatively prime, where G_1 is as defined in the previous theorem. Let G' be the graph obtained from G by adding a path of ℓ edges between x and y, and let G'_1 be the graph obtained from G' by deleting the single edge between any two adjacent vertices in the chain. Then $|K(G_1)|$ and $|K(G'_1)|$ are relatively prime. In particular, it follows from Theorem 2.6 that $K(G')$ is cyclic.*

We note the similarities between Theorem 2.7 and Theorem 2.4. This leads us to pose the following question.

Open Question 2.8. Given a graph G and a pair of vertices x, y so that $|K(G)|$ and $|K(G_1)|$ are relatively prime, must it be the case that the configuration $\delta_{x,y}$ is a generator of $K(G)$?

3 Recurrence Relations and Orders of Critical Groups

Given a finite list of integers $k_1, \ldots, k_n$ with all $k_i > 1$, we define G_n to be a stack of polygons $\mathcal{P}_1, \ldots, \mathcal{P}_n$ where $\mathcal{P}_i$ is a k_i-gon, and $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ share the edge denoted by e_i. Such a graph is not uniquely defined by the n-tuple, as we could stack the polygons along different edges and get different graphs. However, we will see in this section that all such graphs will have the same critical group. In particular, it follows from Theorem 2.4 that $K(G_n)$ is a cyclic group. Moreover, it is a consequence of the Matrix Tree Theorem that the order of the critical group of any graph is equal to the number of spanning trees of the graph, so this number will fully determine $K(G_n)$.

In order to count spanning trees on our polygonal graphs, we use a variant on the technique of deletion-contraction which was developed by Tutte in the 1940's after reading some ideas of Kirchhoff related to electronic resistances. In essence, this method relates the Tutte polynomial of a graph to the Tutte polynomials of the graphs that are obtained by choosing an edge and either deleting it or contracting it. One can then use the fact that the number of spanning trees is the evaluation of the Tutte polynomial at specific values. Rather than rely on this full machinery, our discussion will be self-contained, but we refer the interested reader to [15] for a description of the history of these ideas and [1] for further technical details.

Theorem 3.1. *Setting $T(k_1, \dots, k_n)$ to be the number of spanning trees on the graph G_n we have the following recurrence relation:*

$$T(k_1, \dots, k_n) = k_n T(k_1, \dots, k_{n-1}) - T(k_1, \dots, k_{n-2}).$$

Proof. Let $\mathbb{T}_n$ (resp. $\mathbb{T}_{n-1}, \mathbb{T}_{n-2}$) denote the set of spanning trees on G_n (resp. G_{n-1}, G_{n-2}). In the discussion that follows, we will think of a spanning tree $\mathcal{T}$ of a graph G as being the set of edges in the tree. It is also useful to note that a set of edges on a graph G will be a spanning tree if and only if it consists of exactly $|V(G)| - 1$ edges, at least one of which is adjacent to every vertex of the graph.

We define a map $\Phi : \mathbb{T}_n \cup \mathbb{T}_{n-2} \to \mathbb{T}_{n-1}$ in the following way.

- If $\mathcal{T} \in \mathbb{T}_{n-2}$ we let $\Phi(\mathcal{T}) = \mathcal{T} \cup (\mathcal{P}_{n-1} \setminus \{e_{n-2} \cup e_{n-1}\})$.
- If $\mathcal{T} \in \mathbb{T}_n$ and $(\mathcal{P}_n \setminus \{e_{n-1}\}) \subseteq \mathcal{T}$ then we let $\Phi(\mathcal{T}) = (\mathcal{T} \setminus \mathcal{P}_n) \cup \{e_{n-1}\}$.
- If $\mathcal{T} \in \mathbb{T}_n$ and $\mathcal{T}$ does not contain all of $(\mathcal{P}_n \setminus \{e_{n-1}\})$ then we define $\Phi(\mathcal{T}) = \mathcal{T} \setminus (\mathcal{P}_n \setminus \{e_{n-1}\})$.

One can check that for each $\mathcal{T}$ we have that $\Phi(\mathcal{T})$ will be a spanning tree of G_{n-1}. In particular, we note that in the first case one is adding both $k_{n-1} - 2$ edges and vertices as one moves from G_{n-2} to G_{n-1}. Similarly, in the latter two cases one is removing both $k_n - 2$ edges and vertices as one moves from G_n to G_{n-1}. Examples of this map for trees on the graph G_3 consisting of a stack of three squares is given in Figure 5.

If $\mathcal{T}'$ is a spanning tree of G_{n-1} so that $e_{n-1} \in \mathcal{T}'$ then one can see that there are k_n trees $\mathcal{T} \in \mathbb{T}_n$ so that $\Phi(\mathcal{T}) = \mathcal{T}'$. In particular, the preimages of $\mathcal{T}'$ are exactly the trees $(\mathcal{T}' \setminus \{e_{n-1}\}) \cup (\mathcal{P}_n \setminus \{f_i\})$, as the f_i ranges over all k_n edges of $\mathcal{P}_n$. On the other hand, if $\mathcal{T}'$ is a spanning tree of G_{n-1} so that $e_{n-1} \notin \mathcal{T}'$ then there will be $k_n - 1$ elements of $\mathbb{T}_n$ which map to $\mathcal{T}'$ (in particular, the trees $\mathcal{T}' \cup (\mathcal{P}_n \setminus \{e_{n-1}, f_i\})$ as f_i ranges over the edges of $\mathcal{P}_n$ other than e_{n-1}) and there is a single tree $\mathcal{T} \in \mathbb{T}_{n-2}$ so that $\Phi(\mathcal{T}) = \mathcal{T}'$, namely $\mathcal{T}' \setminus (\mathcal{P}_{n-1} \setminus \{e_{n-2}\})$.

Combining these cases shows that the map Φ is both surjective and k_n-to-1. This implies the theorem. □

Example 3.2. Let us consider the case where we have a stack of k-gons with $k \geq 2$, and let T_n be the number of spanning trees of such a graph so that the critical group of this graph is isomorphic to $\mathbb{Z}/T_n\mathbb{Z}$. In particular, this will be the case where k_n is the constant value k for all n, so Theorem 3.1 implies that the sequence $\{T_n\}$ satisfies the second order linear recurrence $T_n = kT_{n-1} - T_{n-2}$. One can easily compute the initial conditions $T_0 = 1$ and $T_1 = k$.

If one prefers an explicit formula to a recursive one, it is then possible to use well-known results on recurrence relations (see, for example, [14, Ch. 6]) to compute

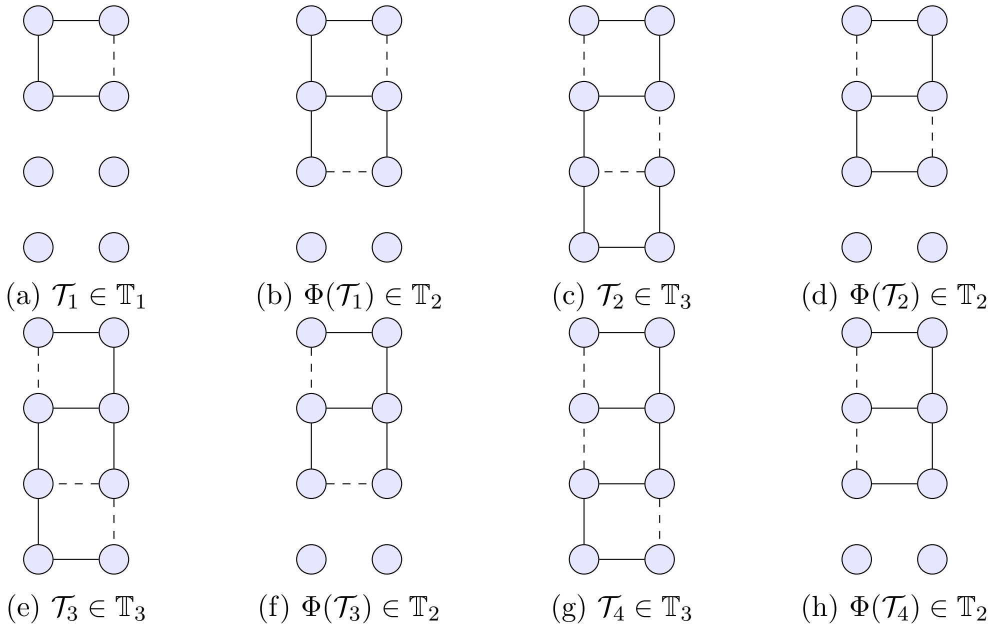


Figure 5: Examples of the map $\Phi : \mathbb{T}_1 \cup \mathbb{T}_3 \to \mathbb{T}_2$

that if $k = 2$ we have that $T_n = n + 1$ and if $k \geq 3$ then we have

$$T_n = \frac{1}{2}\left[\left(1 + \frac{k}{\sqrt{k^2-4}}\right)\left(\frac{k+\sqrt{k^2-4}}{2}\right)^n + \left(1 - \frac{k}{\sqrt{k^2-4}}\right)\left(\frac{k-\sqrt{k^2-4}}{2}\right)^n\right]$$

It is worth noting that when $k = 4$, the graph G_n is the 2-by-n grid and the number of spanning trees is computed in [7] using similar techniques to ours. Moreover, in the case of $k = 3$ our result gives the same answer obtained in [5] by different methods. Finally, in the case where $k = 2$ our graph is the 'banana graph' consisting of two vertices connected by $n + 1$ edges, in which case it is well known that the critical group is $\mathbb{Z}/(n + 1)\mathbb{Z}$.

Example 3.3. Next, consider the example of an n-story 'house', corresponding to the $(n + 1)$-tuple $(3, 4, \ldots, 4)$. As in the previous example, the number of trees will satisfy the recurrence relation $T_n = 4T_{n-1} - T_{n-2}$. One can compute by hand in this case that $T_0 = 3$ and $T_1 = 11$. In particular, this shows that

$$T_n = \frac{1}{2\sqrt{3}}\left[\left(3\sqrt{3}+5\right)\left(2+\sqrt{3}\right)^n + \left(3\sqrt{3}-5\right)\left(2-\sqrt{3}\right)^n\right]$$

Example 3.4. For our final example, we consider the case of a stack of alternating k_1-gons and k_2-gons, where k_1 and k_2 are both at least 2. We further assume that we are not in the case where $k_1 = k_2 = 2$ in order to simplify the calculations. Again, it follows from Theorem 2.4 that the critical group is cyclic and therefore we only need to count the number of spanning trees to determine the group. Let us assume that A_n is the number of spanning trees of the graph formed by adding n of each type of shape in an alternating fashion. (We leave as an exercise to the reader the interesting fact that you get a different answer if you put a stack of n k_1-gons on top of a stack of n k_2-gons). Moreover, let B_n be the number of spanning trees of a graph composed with n k_1-gons and $n-1$ k_2-gons arranged alternatingly.

In particular, it follows from Theorem 3.1 that we have $A_n = k_2B_n - A_{n-1}$ and $B_n = k_1A_{n-1} - B_{n-1}$. From these two relations, one can deduce that $A_n = (k_1k_2-2)A_{n-1}-A_{n-2}$ and $B_n = (k_1k_2-2)B_{n-1}-B_{n-2}$. Combined with the additional observations that $A_0 = 1$, $A_1 = k_1k_2 - 1$, $B_0 = 0$, and $B_1 = k_1$ one can use standard results on recurrence relations to get the following explicit formulas for the A_n and B_n:

$$A_n = \left(\frac{\sqrt{\omega}+\gamma}{2\sqrt{\omega}}\right) \cdot \left(\frac{\gamma-2+\sqrt{\omega}}{2}\right)^n + \left(\frac{\sqrt{\omega}-\gamma}{2\sqrt{\omega}}\right) \cdot \left(\frac{\gamma-2-\sqrt{\omega}}{2}\right)^n$$

$$B_n = \left(k_1+1-\frac{\gamma}{2}\right) \cdot \left(\frac{\gamma-2+\sqrt{\omega}}{2}\right)^n + \left(\frac{\gamma}{2}-k_1-1\right) \cdot \left(\frac{\gamma-2-\sqrt{\omega}}{2}\right)^n$$

where $\gamma = k_1k_2$ and $\omega = \gamma^2 - 4\gamma$.

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