7-22-2016

A Three-Fold Approach to the Heat Equation: Data, Modeling, Numerics

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Spayd, Kimberly, and James Puckett. "A Three-Fold Approach to the Heat Equation: Data, Modeling, Numerics." PRIMUS 26, no. 10 (2016). pp. 938-951.

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Abstract

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Keywords

Mathematical modeling, partial differential equations, student project

Disciplines

Analysis | Mathematics | Physics

A Three-Fold Approach to the Heat Equation: Data, Modeling, Numerics

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Abstract: This article describes our modeling approach to teaching the one-dimensional heat (diffusion) equation in a one-semester undergraduate partial differential equations course. We constructed the apparatus for a demonstration of heat diffusion through a long, thin metal rod with prescribed temperatures at each end. The students observed the physical phenomenon, collected temperature data along the rod, then referenced the demonstration for purposes in and out of the classroom. Here, we discuss the experimental setup, how the demonstration informed practices in the classroom and a project based on the collected data, including analytical and computational components.

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1 INTRODUCTION

Incorporating data and/or numerical experiments in classroom examples of mathematical models is a popular way to motivate student interest and learning. See [1, 3, 4, 6, 7] as a small sample. Moreover, students can gain insight into the application, intuition for the mathematical analysis and confirmation of the theoretical results by using real data and numerical simulations to complement the theory [2]. In these ways, students gain experience in the techniques of applied mathematics by exploring a model along three axes: mathematics, data analysis, and computation [8].

At our institution, explicitly applied courses have not historically been the standard offering in the Math department. Recently, though, there has been a concerted effort to offer, and student interest in, more applied courses. In the past few years, courses have been offered covering applied linear algebra (focused on ranking and clustering methods), operations research, neurobiology models and partial differential equations. It was in preparation for the selected topics course in partial differential equations when we started discussing the importance of linking the PDE models to be discussed with their physical counterparts through real data and numerical simulations.

The upper level selected topics course in partial differential equations was offered for the first time during the Spring 2015 semester. The prerequisite courses were Multivariable Calculus and Ordinary Differential Equations. Of the fifteen students enrolled there were seven seniors, five juniors, and three sophomores. All but two students were either majoring or minoring in mathematics. All but one of the math majors were simultaneously pursuing another major or a minor in various areas, including physics, economics, music, environmental science, computer science and education. When asked why they wanted to take a PDE course, the students' responses focused on the applicability to real-world problems and a change of pace from the pure courses more regularly offered. This affirmed our intent to highlight techniques of applied mathematics; in particular, we wanted to make explicit connections between physical phenomena and their PDE models in setting the context for the mathematics.

One such standard model discussed in a one-semester undergraduate PDE course is the one-dimensional heat equation, also known as the diffusion equation, given by

$$u_t = k u_{xx}, \qquad x \in (0, L), \quad t > 0, \tag{1}$$

where $u = u(x,t)$ is a scalar representing the temperature of a very thin rod at any point in space and time and k is a material-dependent parameter known as thermal diffusivity. The ends of the rod at $x = 0, L$ are subject to various boundary conditions: prescribed temperature, insulation and Newton's Law of Cooling. Traditionally, these are referred to as Dirichlet, Neumann and Robin boundary conditions, respectively. See Figure 1 for a schematic representation with Dirichlet boundary conditions at each end. In the course at our institution, we used Haberman's *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems* as the textbook [5]. The students immediately encountered the derivation of the one-dimensional heat equation, a discussion of boundary conditions and steady-state (time-independent) solution methods in the first chapter.

Figure 1: A very thin rod from $x = 0$ to $x = L$ with prescribed temperature (Dirichlet) boundary conditions $u(0,t) = T_0$ and $u(L,t) = T_L$.

For this reason, we wanted to set the physical context of the one-dimensional heat equation as soon as possible; the students met in a physics laboratory for the second class meeting in order to observe a demonstration of heat diffusing through a thin rod, described in Section 2. In Section 3, we address how the demonstration helped with the model's derivation and solution techniques in the classroom. Temperature data recorded during the demonstration was used in a student project later in the semester; see Section 4 for a description of the project. After the project's completion, the students were asked for feedback on all parts of the experience. The results are summarized in Section 5.

2 EXPERIMENTAL SETUP

We designed a simple and affordable apparatus to demonstrate heat diffusion in the same context as the one-dimensional heat equation. The necessary funds were provided by [an internal organization at our institution]. The apparatus consisted of a thin rod placed between two temperature baths as in Figure 1, with four temperature sensors attached. Using more sensors would have decreased the sampling rate. The rod was made of aluminum, had a length $L = 300$ mm and a square cross section with side length 3.2 mm. The flat sides on the rod allowed for better thermal contact between the temperature sensors and the rod. The temperature sensors (Maxim Integrated DS18B20) were attached to the rod at four locations: $x = 47, 94, 141, 188$ mm. The rod and sensors were encased in a 25 mm thick foam tube used for insulation. These sensors were connected in parallel to a Raspberry Pi which read each sensor's unique serial number and temperature at a rate of 0.3 Hz. Before students arrived in class, all the equipment was placed on the lab table and allowed to reach thermal equilibrium at $T = 24°$C. The heat source consisted of 1 liter of water in a glass beaker placed on a hot plate, which was checked for a stable temperature reading at $T = 53°$C.

After students arrived, we began recording temperatures from the sensors. A python script recorded the data from the sensors and streamed them to the command-line where students could see each sensor's reading plotted against time (Figure 2). The rod was then carefully placed in thermal contact with the room temperature water bath first and then with the hot water bath. More details about the setup, a parts list, the python script used to record and analyze the data, along with the collected temperature readings are available at: website address with author information

3 MODEL BUILDING AND ANALYSIS

In the third class meeting, we went through the formal derivation of the one-dimensional heat equation as an abstraction from what had just been observed. At this point in time, though, the schematic drawing of a thin rod, as in Figure 1, and the analysis that followed were rooted in a common memory. We were explicitly tied to the observed physical experiment.

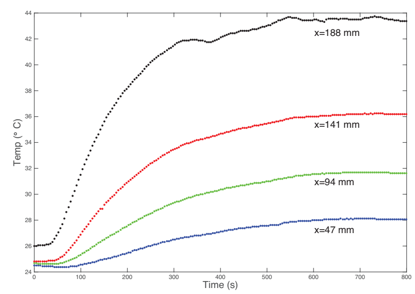


Figure 2: Temperature readings from four sensors along the rod over the course of the heat diffusion demonstration

The physical assumptions needed for the derivation of Equation 1 include the following:

- the rod is perfectly insulated so that no heat loss occurs through the lateral surface of the rod,
- the rod is composed of a single, uniform material so that thermal properties are constant,
- there is no internal heat generation,
- the rod is either heated or cooled only at the ends, not across it.

Because of the laboratory demonstration, these assumptions evolved from passive requirements into discussion points about the model's relevance and accuracy. The students initiated conversations regarding which assumptions were met, neglected or approximated in the lab. This was a natural segue into the role of mathematical models as approximations of reality and the use of data in model validation. As will be discussed in Section 4, this was not an obvious point to the students given that many of them had no prior exposure to rigorous mathematical modeling.

Conservation of energy is the governing equation in the derivation of Equation 1:

$$\frac{d}{dt}\int_a^b c(x)\rho(x)u(x,t)A dx = \phi(a,t)A - \phi(b,t)A + \int_a^b Q(x,t)A dx \quad (2)$$

where $[a, b]$ is an arbitrary subinterval of $[0, L]$, c is the specific heat of the rod $(\mathrm{J}\cdot\mathrm{kg}^{-1}\cdot\mathrm{K}^{-1})$, ρ is the mass density of the rod $(\mathrm{kg}\cdot\mathrm{m}^{-3})$, A is the constant cross-sectional area of the rod, ϕ represents heat energy flux and Q gives the rate of internal heat generation. Under the third bulleted assumption above, $Q(x,t) \equiv 0$. Then the left hand side of

Equation 2 gives the rate of change in the total heat energy and is equal to the difference of heat energy per unit time between endpoints a and b on the right hand side. Given a uniform rod, $c(x)$ and $\rho(x)$ are constants that are incorporated into the diffusivity constant k in Equation 1.

It was significantly easier to motivate, explain and understand Equation 2 within the context of the lab demonstration than without it. Specifically, the above equality between the rate of change of total heat energy in an arbitrary section of the rod and the difference between inflow and outflow rates was physically relevant as opposed to a theoretical postulation. The students had a shared physical reference and it made a strong base on which to build the theory.

Likewise, the boundary conditions discussed after the derivation were entirely accessible after the lab demonstration. The mathematical terminology of Dirichlet, Neumann, and Robin boundary conditions were coupled to the realistic experimental conditions associated with each term: holding the temperature fixed at an end, insulation at an end prohibiting heat loss, and heat escaping through one end into the air, respectively. As with conservation of energy above, the students could visualize these conditions in the context of their lab experience and make more personal connections with the material.

We began investigating solution techniques by first identifying an equilibrium temperature distribution, i.e. a steady-state solution, u_s. During the demonstration, the students observed a "long-run" plateau of temperatures from each temperature sensor, seen in Figure 2. It was clear to them that at some point, the temperature readings became steady and did not depend on time. Thus $u_t = 0$ and Equation 1 reduces to

$$u_{xx} = 0. \tag{3}$$

Additionally, we plotted the equilibrium temperatures from the four temperature sensors (see Figure 3) and, from it, could predict that the steady-state solution would be a linear function of x. The theory supports this empirical observation: integrating Equation 3 twice with Dirichlet boundary conditions $u(0,t) = T_0$ and $u(L,t) = T_L$ gives

$$u_s(x) = \frac{T_L - T_0}{L}x + T_0. \tag{4}$$

The benefits of the lab demonstration also extended into our coverage of separation of variables as a general solution technique. In the examples covered in class and the homework exercises, all boundary conditions were homogeneous, e.g. $u(0,t) = 0, u_t(L,t) = 0$. This differed from the boundary conditions used in the demonstration; however, the students had no trouble extending the observed physical context to a situation in which there was an ice bath at one end of the rod ($u(0,t) = 0$) and insulation at the other ($u_t(L,t) = 0$), for example.

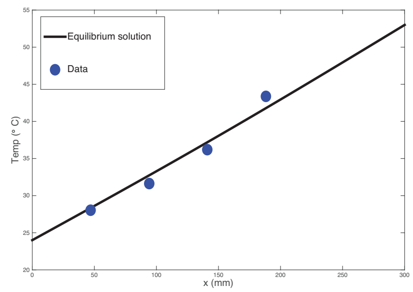


Figure 3: Equilibrium temperatures from the four sensors in the heat diffusion demonstration (dots) and the steady-state solution (line)

4 PROJECT

Apart from the positive effects of writing and group work on student learning in the context of mathematical modeling [8], assigning projects that require a lengthy written report are standard practice in the Math department at our institution. The project included here was assigned in the third week of class. In the following subsections, the numbered questions were given to the students as parts of their project. Figures from the project output are included where appropriate.

4.1 Analysis

Since we intentionally used nonhomogeneous boundary conditions in the lab demonstration, the method of separation of variables could not immediately translate to the physical problem that was actually observed. Chapter 8 in [5], which was not included in the semester-long course, covers the topic of treating the solution to this type of problem as the sum of two parts: the solution of the associated homogeneous initial-boundary value problem,

$$u_t = ku_{xx}, \quad x \in (0, L), \quad t > 0, \tag{5}$$

$$u(0, t) = 0, \quad t > 0, \tag{6}$$

$$u(L, t) = 0, \quad t > 0, \tag{7}$$

$$u(x, 0) = f(x), \quad x \in (0, L), \tag{8}$$

and the steady-state solution, u_s, to the original nonhomogeneous problem. This approach was entirely accessible to the students and appropriate for independent study after the first few weeks of the semester; it followed as a natural extension of our coverage of separation of variables with homogeneous boundary conditions.

To guide the students through this material, it was broken into smaller pieces as seen in the list of questions below.

1. What is the initial boundary value problem for the heat equation corresponding to the experimental setup you observed in the lab?
2. What is the steady-state solution, $u_s(x)$, to the model you wrote above?
3. Follow the steps in your notes and/or section 2.3 in the book to make an attempt at solving for $u(x,t)$. What problem do you encounter? Why?
4. Instead of solving for $u(x,t)$, consider the function $v(x,t) = u(x,t) - u_s(x)$, which measures the displacement of the time-dependent solution from the steady-state solution. What is the associated initial boundary value problem for $v(x,t)$?
5. Solve for $v(x,t)$. Why do you not have the same trouble as you did in the third question? Now, what is the solution $u(x,t)$ of the original heat equation model from the first question?

The first two questions are perfunctory but necessary for setting up the more involved analysis in questions three through five. In particular, the students needed to carefully analyze the logic in the separation of variables technique with nonhomogeneous boundary conditions. With homogeneous boundary conditions, it can be assumed that the spatial factor in the separated solution is zero at the boundaries, so that we avoid the trivial solution in the temporal factor, and hence full solution. This is not a valid assumption when the boundary conditions are nonhomogeneous. This was a subtle point that many students had difficulty identifying without help. Once this was accomplished, though, the students were then able to proceed without much difficulty to determine that the solution is the sum of an infinite series ($v(x,t)$) and the steady-state solution ($u_s(x)$).

4.2 Numerics

In class, we spent a significant amount of time discussing the Matlab code used to visualize the first n terms of a Fourier series solution of a given initial boundary value problem. In order to answer the first question below, students needed to adjust provided code from class to this specific context. For the second and third questions below, the students were expected to make use of Matlab's help features and write the necessary code themselves.

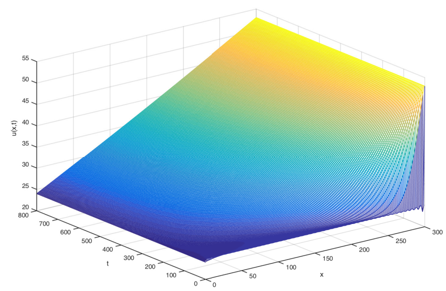
1. Use Matlab to graph the series solution you obtained above for $u(x,t)$.

2. Use Matlab's PDE solver **pdepe** to simulate the solution to the full one-dimensional heat equation problem corresponding to what you observed in the lab. How does this numerical simulation compare to the plot of your analytic solution, particularly near the initial condition? What causes the difference between the two plots?
3. Overlay the four data points from the experiment on the simulations from the previous question. Discuss how well (or not) the experimental data points match up with the numerical solution. Compare the data and model over time.

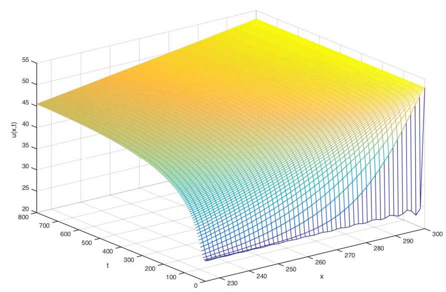
A week before the project due date, we used one class period as a question and answer session specifically for Matlab issues. Students were asked on an anonymous mid-semester evaluation whether they thought the expectations for Matlab programming in the first project were appropriate; responses were mixed but it was clear that the provided individual and group help sessions were appreciated.

As a foreshadow of the Gibbs phenomenon, covered later in the semester, students were asked in the second question above to compare the truncated analytical solution and the full numerically simulated solution. See Figure 4, parts (a) and (c), for the two surface plots. Small oscillations can been seen in the bottom-right-front corner of Figure 4(a), where t and u are small and x is large. Figure 4(b) is a closer look at this behavior. Students were able to answer correctly that necessary truncation of the infinite series was to blame for these oscillations. This reflection came in handy several weeks later when the Gibbs phenomenon was discussed in class.

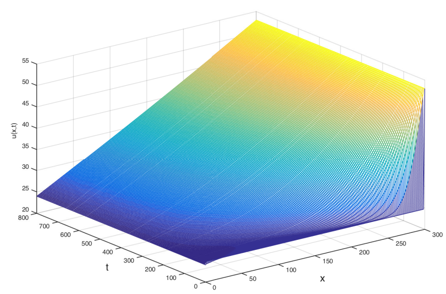
The comparison of the temperature data and numerical simulation in the third question was the most enlightening in regard to mathematical modeling. It was through this particular visualization (Figure 5) that the students were best able to appreciate the heat equation model as an abstraction of the physical phenomenon. Several students wrote in their project reports that the heat equation solution was not very accurate when compared to the data because they did not align closely enough, as in Figure 5(b),(c). This led to a class discussion about unavoidable discrepancies between real data and theoretical models; our primary hope is that a mathematical model at least captures the qualitative behavior of a physical phenomenon and perfect alignment between real data and the theoretical solution is not necessarily expected. These points were well received but not obvious to everyone since many of the students had no prior experience using data in this way.



(a) Truncated analytical solution



(b) Closer look at oscillation in (a)

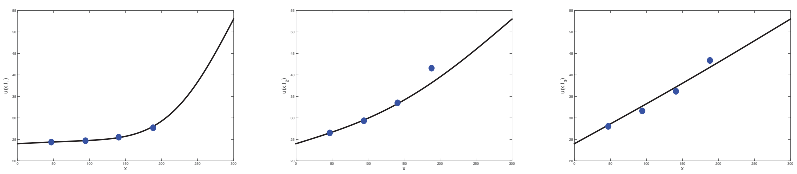


(c) Numerical simulation

Figure 4: Plots used in comparing the analytic solution and numerical simulation in the second question from Section 4.2

4.3 Errors and Fine-Tuning

The last two questions in the project were written to highlight the limits of our mathematical model, partially addressed in the previous sec-



(a) $t_1 = 60.3379$ s (b) $t_2 = 302.1575$ s (c) $t_3 = 799.3076$ s

Figure 5: Collected temperature data (dots) and numerical simulation (curve) at times t_1, t_2, t_3; these are snapshots of the animation produced for the third question in Section 4.2

tion.

1. What are some sources of error between the physical experiment and mathematical model? Explain.
2. What parameters could be changed to fine-tune the model, so that the model better matches the data points? Experiment with parameter values in the Matlab code to see if you can get the model to fit the data any better.

Several responses were insightful, particularly from the physics majors in the class; students identified measurement inaccuracies from the temperature sensors, imperfect insulation and variable boundary conditions as possible sources of error. The thermal diffusivity constant was easily identified as a tuning parameter. Although the thermal diffusivity of aluminum is 8.4×10^{-5} $\mathrm{m}^2 \cdot \mathrm{s}^{-1}$, we found the best fit for the data with $k = 2.6 \times 10^{-5}$. This can be explained by thermal loss due to imperfect insulation. Some students went further than simply refining the value of k and attempted to incorporate time-dependent boundary conditions and a space-dependent thermal diffusivity function.

5 REFLECTIONS

To measure student perceptions of the demonstration, associated coursework and project, a short anonymous evaluation was given after the project reports were graded and returned. The students were asked to give a numerical measure between one and five to each statement included in the evaluation, where the value one represented strong disagreement, three represented neutrality and five represented strong agreement. All fifteen students enrolled in the course completed the evaluation.

The statements and mean responses are presented in Table 1. The distributions of responses were not significantly skewed, so mean and median values were similar. The lowest responses were in regard to the derivation of the heat equation (the second statement in Table 1); it is likely that a substantial portion of the math majors in the class felt a bit out of their depth with the heavily applied derivation. Even in the face of this, the responses to this statement are still generally positive.

Statement	**Mean Agreement**
The demonstration added to my understanding of the *physical context* for the heat equation.	4.47
The demonstration added to my understanding of the *derivation* of the heat equation model in class.	3.93
The demonstration added to my understanding of the various *boundary conditions* discussed in class.	4.27
The project added to my appreciation of applied mathematics and mathematical modeling.	4.27
The demonstration, resulting data and analysis enhanced my educational experience in this class.	4.67

Table 1: Statements and mean responses from student evaluations

Prior to giving the evaluations, we had hoped that the responses would at least center around the value four. This would signify to us that the students perceived a positive impact on their learning through the lab experience. We were pleased to see that the responses were very favorable. All of the written comments were positive; one student noted that this project helped their "growth as a mathematician."

Overall, we believe this entire experience was greatly beneficial for the students. It was clear that the students enjoyed being in the lab for the demonstration; it was a camaraderie-building experience and an active component in an otherwise traditional math class. It appeared to us that the novelty and physicality of the demonstration increased the students' enthusiasm for the material. This naturally led to more engagement, discussion and insight than we would have expected otherwise. This approach to the heat equation, from the lab to the project write-up, allowed the students to experience mathematics as an applied and interdisciplinary field. The data, modeling and numerical components all worked together to significantly enhance the learning that took place.

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