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Abstract

We consider how a standard theorem in algebraic geometry relating properties of a curve with a $(\mathbb{Z}/2\mathbb{Z})^2$ -action to the properties of its quotients generalizes to results about sets and graphs that admit $(\mathbb{Z}/2\mathbb{Z})^2$ -actions.

Keywords

Mathematical theorems, Quotients, Vertices, Jacobians, Algebra, Genera, Kanis, Geometry, Curves, Graph Theory

Disciplines

Algebraic Geometry | Mathematics

Klein Four Actions on Graphs and Sets

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Abstract

We consider how a standard theorem in algebraic geometry relating properties of a curve with a $(\mathbb{Z}/2\mathbb{Z})^2$ -action to the properties of its quotients generalizes to results about sets and graphs which admit $(\mathbb{Z}/2\mathbb{Z})^2$ -actions.

In studying the algebraic geometry of curves that admit automorphisms, one is led to see that properties of the curve are often related to properties of its quotients. To see one example of such a theorem, let $K = \{id, \sigma_1, \sigma_2, \sigma_3\}$ be the Klein-Four group $(\mathbb{Z}/2\mathbb{Z})^2$, so that id is the identity element, $\sigma_i^2 = id$ and $\sigma_i\sigma_j = \sigma_k$ for any permutation of the three involutions. For notational convenience, we define the subgroups $K_i = \{id, \sigma_i\}$. In that case, we have the following result.

Theorem 1. *Let X be an algebraic curve so that K acts on X . Define the quotient curves $X_i = X/K_i$ and $X_0 = X/K$. Then we have the relationship between the genera of the curves:*

$$g(X) + 2g(X_0) = g(X_1) + g(X_2) + g(X_3).$$

In this note, we show that analogous results hold if one considers sets or graphs that come equipped with a K -action. We begin by considering the case of finite sets. Recall that if a group G acts on a set S , then we define the quotient set G/S to be the set of orbits of elements of S under the G -action.

Theorem 2. *Let S be any finite set that is equipped with a K -action and define the quotient sets $S_i = S/\langle\sigma_i\rangle$ and $S_0 = S/K$. Then we have the following relationship among the sizes of the sets:*

$$|S| + 2|S_0| = |S_1| + |S_2| + |S_3|.$$

In order to prove this theorem, we will use the following Lemma.

Lemma 3. *Assume the finite group G acts on the set T . For each element $g \in G$, let T^g be the set of elements of T fixed by g . Then $|T/G| = \frac{1}{|G|} \sum_{g \in G} |T^g|$.*

This lemma is most commonly referred to as Burnside's Lemma, although Burnside himself attributed it to Frobenius and others have attributed it to Cauchy. For details about the history of this lemma, as well as two separate proofs, we refer the reader to [7].

Proof of Theorem 2. Applying Lemma 3 to our group actions, we compute for $i > 0$ that

$$|S_i| = |S/K_i| = \frac{|S^{id}| + |S^{\sigma_i}|}{2} = \frac{|S| + |S^{\sigma_i}|}{2}$$

and similarly

$$|S_0| = \frac{|S| + |S^{\sigma_1}| + |S^{\sigma_2}| + |S^{\sigma_3}|}{4}.$$

Combining these, we see:

$$\begin{aligned} |S_1| + |S_2| + |S_3| &= \frac{3|S| + |S^{\sigma_1}| + |S^{\sigma_2}| + |S^{\sigma_3}|}{2} \\ &= |S| + \frac{|S| + |S^{\sigma_1}| + |S^{\sigma_2}| + |S^{\sigma_3}|}{2} \\ &= |S| + 2|S_0| \end{aligned}$$

proving Theorem 2. □

Remark 4. We note that, with a bit more bookkeeping, one can generalize the proof of Theorem 2 to prove that if D_n is any dihedral group generated by two involutions σ_1 and σ_2 , then $|S| + 2|S/\langle \sigma_1, \sigma_2 \rangle| = |S/\langle \sigma_1 \rangle| + |S/\langle \sigma_2 \rangle| + |S/\langle \sigma_1 \cdot \sigma_2 \rangle|$.

We believe that Theorem 2 is of interest on its own and may have other applications, but our main interest comes from an application to graph theory about the genus of a graph that admits a K -action. In particular, we will consider graphs G which are connected and have no loops or multiple edges. While there are several notions of the genus of a graph, in this note we are referring to the combinatorial genus, also known as the cyclomatic number or the circuit-rank. Explicitly, this can be calculated as the rank of the first Betti homology group, so that $g(G) = |E| - |V| + 1$ for any connected graph G with edge-set E and vertex-set V .

A K -action on a graph is defined by actions of the group K on both the vertex set and the edge set of G . Moreover, these actions must be compatible in the sense that if e is an edge between two vertices v_1 and v_2 , then $\sigma_i \cdot e$ is an edge between the vertices $\sigma_i \cdot v_1$ and $\sigma_i \cdot v_2$. We define the quotient of a graph G by a K -action to be the graph G/K whose vertices are in bijection with the K -orbits of vertices of G and whose edges are given by the K -orbits of edges between vertices in different K -orbits. In particular, note that a quotient will remove any edges between two vertices that lie in the same orbit rather than lead to a loop. This is what Baker and Norine define in [1] to be ‘vertical ramification’ and we will say that an edge is contracted by a group action if its two end points lie in the same orbit. See Figure 1 for an example. In this example, K acts on the graph G by letting σ_1 reflect the graph in the horizontal axis and σ_2 reflect the graph in the vertical axis.

We now prove the following analogy to Theorem 1.

Theorem 5. Let G be a graph without multiple edges so that K acts on G . Define the quotient graphs $G_i = G/K_i$ and $G_0 = G/K$ as above. Then we have the relationship between the genera of the graphs:

$$g(G) + 2g(G_0) = g(G_1) + g(G_2) + g(G_3).$$

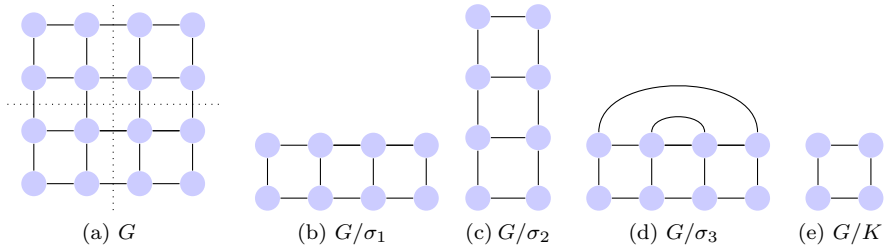


Figure 1: A graph G with K -action and its various quotients

Proof. We begin by looking at the vertex sets V, V_0, V_i of the graphs G, G_0, G_i , ignoring the edges. In particular, we note that K acts on the set V , so it follows directly from Theorem 2 that $|V| = |V_1| + |V_2| + |V_3| - 2|V_0|$.

We next consider the action of K on the set of edges E . As above, we can apply Theorem 2 to the set E in order to get that $|E| + 2|E/K| = |E/K_1| + |E/K_2| + |E/K_3|$. In this case, however, the number of elements of the quotient sets $|E/K|$ and $|E/K_i|$ is not necessarily the same as the number of edges of the quotient graphs G/K and G/K_i , because some edges may be contracted. In particular, let us define the numbers $c_i = |E/K_i| - |E_i|$ for $i = 1, 2, 3$ and $c_0 = |E/K| - |E_0|$, so that each c_i gives the number of edges contracted by the relevant action.

Consider an edge e whose endpoints u and v are in the same K -orbit, so that e is contracted by the K -action. In particular, there must be some σ_i which interchanges u and v . Because our graph does not have multiple edges this action must fix e . Thus, the subgroup of K consisting of all elements that fix e has either order 2 or 4, and we consider these two cases separately.

If all of the σ_i fix a given edge e , then the fact that at least one of the σ_i switches the endpoints implies that exactly two of them do. Therefore, e is contracted by exactly two of the K_i -actions, meaning that this orbit contributes two to the sum $c_1 + c_2 + c_3$.

On the other hand, if only one of the σ_i fixes the edge e , then without loss of generality we may assume that $\sigma_1(e) = e$ and σ_2 and σ_3 each interchange e with a different edge f . We denote the endpoints of e as u_1 and u_2 and the endpoints of f as v_1 and v_2 where $\sigma_2(u_i) = v_i$ and vice versa. By our assumption that K contracts e , it must be the case that $\sigma_1(u_1) = u_2$ and $\sigma_1(u_2) = u_1$. It follows that $\sigma_3(v_1) = \sigma_1(\sigma_2(v_1)) = \sigma_1(u_1) = u_2$ and thus that $\sigma_1(v_1) = \sigma_2(\sigma_3(v_1)) = \sigma_2(u_2) = v_2$. Similarly, $\sigma_1(v_2) = v_1$. In particular, we see that both e and f are contracted by the group action K_1 , but neither are contracted by K_2 or K_3 , so that this orbit also contributes two to the sum $c_1 + c_2 + c_3$.

Because all edges that are contracted by K contribute two to the sum, it follows that $c_1 + c_2 + c_3 = 2c_0$ and therefore that $|E| = |E_1| + |E_2| + |E_3| - 2|E_0|$.

Given that $g(G) = |E| - |V| + 1$, the proof of the theorem is now clear. \square

We note that one can relax the restriction that G has multiple edges somewhat, but we cannot remove it entirely. In particular, consider the graph \mathcal{G} in Figure 2 along with the K -action where σ_1 permutes the edges as $(ab)(cd)$ while leaving the vertices fixed and σ_2 permutes the edges as $(ad)(bc)$ while also switching the two vertices. One can easily check that in this case the quotient graphs $\mathcal{G}_2, \mathcal{G}_3$, and \mathcal{G}_0 all consist only of a single point while \mathcal{G}_1 has genus one. However, the genus of \mathcal{G} is 3, making the conclusion of the theorem incorrect. However, one can show that it is only in situations like this that the conclusion of Theorem 5 fails; in particular, it will be true as long as the graph G does not contain \mathcal{G} as a subgraph with an action such as this one, even if it does contain other situations with multiple edges.

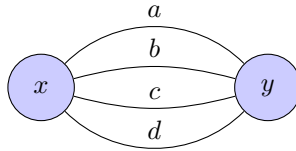


Figure 2: A counterexample \mathcal{G} to the conclusion of Theorem 5

Theorem 5 gives a discrete analogue to Theorem 1, and is one of a family of results translating theorems in algebraic geometry to theorems in graph theory and vice versa. For more examples, see [1] and [2]. It is worth noting that in the case where the base field does not have characteristic two, Theorem 1 can be proven directly from Theorem 2 by considering the branch points of the various $(\mathbb{Z}/2\mathbb{Z})$ -covers and using the Riemann-Hurwitz formula that compares the genera of the curves in a cover $X \rightarrow Y$; for more details of these constructions we refer the reader to [4]. In [1], Baker and Norine prove a Graph Theoretic analog of the Riemann-Hurwitz formula which compares the genera of graphs in a cover $G \rightarrow H$ and under additional hypotheses this can be used to give an alternate proof of Theorem 5.

On the other hand, Theorem 1 is also an immediate consequence of a theorem due to Kani and Rosen [6], which proves the much stronger statement that $\text{Jac}(X) \oplus (\text{Jac}(X_0))^2$ is isogenous to $\bigoplus_i \text{Jac}(X_i)$. Theorem 1 is then a consequence of the fact that the genus of a curve is the dimension of its Jacobian. Given the analogy between Jacobians of graphs and Jacobians of curves developed in the papers listed above, one is naturally led to ask whether a similar result holds about the Jacobians of graphs. If one assumes the additional restriction that the group action is harmonic, meaning that if any group element fixes an edge, then it must switch the endpoints, then one is able to prove explicit results about the Jacobians of the graphs in terms of the Jacobians of the quotient graphs, as shown in [3] and [5]. However, in the case of graphs the

relationship between the genus and the Jacobian is more subtle than in the case of curves, so Theorem 5 is not an immediate consequence of these results.

The result of Kani and Rosen is more general than is stated above, and applies to any curve that has an action of a group G which can be written as the union of subgroups which are ‘almost disjoint’, meaning that any two of the subgroups intersect only in the identity element. The dihedral groups D_n form one such family of groups, and the results in [5] show that under certain additional hypotheses one can decompose the Jacobian of any graph with a dihedral action in terms of the Jacobians of its quotients. As discussed in Remark 4, several of the key ideas in our proof of Theorem 5 generalize to dihedral groups, suggesting that one might be able to generalize the theorem to this family or to other groups satisfying Kani and Rosen’s hypotheses. We leave this for future authors to explore.

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